Non-sequential weak supercyclicity and hypercyclicity

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arXiv:1209.1462v1 [math.FA] 7 Sep 2012

Abstract

A bounded linear operator T acting on a Banach space $\mathcal{B}$ is called weakly hypercyclic if there exists $x \in \mathcal{B}$ such that the orbit $\{T^n x : n = 0, 1, \dots\}$ is weakly dense in $\mathcal{B}$ and T is called weakly supercyclic if there is $x \in \mathcal{B}$ for which the projective orbit $\{\lambda T^n x : \lambda \in \mathbb{C},\ n = 0, 1, \dots\}$ is weakly dense in $\mathcal{B}$. If weak density is replaced by weak sequential density, then T is said to be weakly sequentially hypercyclic or supercyclic respectively. It is shown that on a separable Hilbert space there are weakly supercyclic operators which are not weakly sequentially supercyclic. This is achieved by constructing a Borel probability measure μ on the unit circle for which the Fourier coefficients vanish at infinity and the multiplication operator $Mf(z) = zf(z)$ acting on $L_2(\mu)$ is weakly supercyclic. It is not weakly sequentially supercyclic, since the projective orbit under M of each element in $L_2(\mu)$ is weakly sequentially closed. This answers a question posed by Bayart and Matheron. It is proved that the bilateral shift on $\ell_p(\mathbb{Z})$, $1 \leqslant p < \infty$, is weakly supercyclic if and only if $2 < p < \infty$ and that any weakly supercyclic weighted bilateral shift on $\ell_p(\mathbb{Z})$ for $1 \leqslant p \leqslant 2$ is norm supercyclic. It is also shown that any weakly hypercyclic weighted bilateral shift on $\ell_p(\mathbb{Z})$ for $1 \leqslant p < 2$ is norm hypercyclic, which answers a question of Chan and Sanders.

1 Introduction

As usual $\mathbb{C}$ and $\mathbb{R}$ are the fields of complex and real numbers respectively, $\mathbb{Z}$ is the set of integers, $\mathbb{N}$ is the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let T be a bounded linear operator acting on a complex Banach space $\mathcal{B}$. An element $x \in \mathcal{B}$ is called a *weakly hypercyclic vector for* T if the orbit

$$O(T, x) = \{T^n x : n \in \mathbb{N}_0\}$$

is weakly dense in $\mathcal{B}$ and T is said to be *weakly hypercyclic* if it has a weakly hypercyclic vector. Similarly $x \in \mathcal{B}$ is called a *weakly supercyclic vector for* T if the projective orbit

$$O_{\rm pr}(T, x) = \{\lambda T^n x : n \in \mathbb{N}_0,\ \lambda \in \mathbb{C}\}$$

is weakly dense in $\mathcal{B}$ and T is said to be *weakly supercyclic* if it has a weakly supercyclic vector.

These classes of operators are more general than the classes of hypercyclic and supercyclic operators, in which the density is required with respect to the norm topology, see the surveys [20] and [18] and references therein and [9, 21, 11, 26, 27] for other related results on weak hypercyclicity and supercyclicity. Weakly supercyclic and weakly hypercyclic operators, although

more general than the supercyclic and hypercyclic ones, enjoy many of the properties of supercyclic and hypercyclic operators. For instance, if T is weakly supercyclic or hypercyclic, then so is T^n for any $n \in \mathbb{N}$. The norm topology version of the latter result was proved by Ansari [1] and the same proof works for weakly supercyclic and hypercyclic operators. Another instance: the operator $\alpha I \oplus T : \mathbb{C} \oplus \mathcal{B} \to \mathbb{C} \oplus \mathcal{B}$, where $\mathcal{B}$ is a Banach space and $\alpha \neq 0$, is supercyclic if an only if $\alpha^{-1}T$ is hypercyclic, see [15]. Again, the proof also works if the norm topology is replaced by the weak one, see [20] and [26]. This observation provides the first known examples of weakly supercyclic non-supercyclic operators on a Hilbert space [26].

Recall that a subset A of a topological space X is called *sequentially closed* if for any convergent in X sequence of elements of A, the limit belongs to A. The minimal sequentially closed set $[A]_{\mathrm{s}}$ containing a given set A (=the intersection of all sequentially closed sets, containing A) is called the *sequential closure* of A. Finally $A \subset X$ is called sequentially dense in X if $[A]_{\mathrm{s}} = X$. Note that in general $[A]_{\mathrm{s}}$ may be bigger than the set of limits of converging sequences of elements of A.

An interesting example in the Hilbert space setting was recently provided by Bayart and Matheron [6]. They proved that if μ is a continuous Borel probability measure on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, supported on a Kronecker compact set, then the multiplication operator $Mf(z) = zf(z)$ acting on $L^2(\mu)$ is weakly supercyclic. On the other hand, since M is an isometry, it cannot be supercyclic, see [2]. It should be noted that in the last example there is x in $\mathcal{B}$ such that any vector from $L_2(\mu)$ is a limit of a weakly convergent sequence of elements of $O_{\mathrm{pr}}(M, x)$.

The last observation motivates the following definitions. A vector $x \in \mathcal{B}$ is called *a weakly sequentially hypercyclic* vector for T if the orbit $O(T, x)$ is weakly sequentially dense in $\mathcal{B}$ and T is called *weakly sequentially hypercyclic* if it has weakly sequentially hypercyclic vectors. A vector $x \in \mathcal{B}$ is called *a weakly sequentially supercyclic* vector for T if the projective orbit $O_{\mathrm{pr}}(T, x)$ is weakly sequentially dense in $\mathcal{B}$ and T is called *weakly sequentially supercyclic* if it has weakly sequentially supercyclic vectors.

Slightly different concepts were introduced by Bes, Chan and Sanders [7] and implicitly by Bayart and Matheron [6]. In fact, they call the following properties weak sequential hypercyclicity and weak sequential supercyclicity. We call them in a bit different way in order to distinguish from the above defined ones. Namely, T is called *weakly* 1*-sequentially hypercyclic* if there exists $x \in \mathcal{B}$ such that any vector from $\mathcal{B}$ is a limit of a weakly convergent sequence of elements of the orbit $O(T, x)$ and T is called *weakly* 1*-sequentially supercyclic* if there exists $x \in \mathcal{B}$ such that any vector from $\mathcal{B}$ is a limit of a weakly convergent sequence of elements of the projective orbit $O_{\mathrm{pr}}(T, x)$.

The obvious relations between the above properties are summarized in the following diagram:

$$
\begin{array}{ccccc}
\boxed{\text{hypercyclicity}} & \Longrightarrow & \boxed{\text{supercyclicity}} & & \\
\Downarrow & & \Downarrow & & \\
\boxed{\text{weak 1-sequential hypercyclicity}} & \Longrightarrow & \boxed{\text{weak 1-sequential supercyclicity}} & & \\
\Downarrow & & \Downarrow & & \\
\boxed{\text{weak sequential hypercyclicity}} & \Longrightarrow & \boxed{\text{weak sequential supercyclicity}} & & \\
\Downarrow & & \Downarrow & & \\
\boxed{\text{weak hypercyclicity}} & \Longrightarrow & \boxed{\text{weak supercyclicity}} & \Longrightarrow & \boxed{\text{cyclicity}}
\end{array}
$$

Bayart and Matheron [6] raised the two following questions.

QUESTION **1**. *Does there exist a bounded linear operator, which is weakly supercyclic and not weakly* 1*-sequentially supercyclic?*

QUESTION **2**. *Does there exist a positive Borel measure μ on $\mathbb{T}$ such that the Fourier coefficients of μ vanish at infinity and the operator $Mf(z) = zf(z)$ acting on $L_2(\mu)$ is weakly supercyclic?*

In view of the following proposition, an affirmative answer to the second question implies an affirmative answer to the first one in the Hilbert space setting.

PROPOSITION **1.1.** *Let μ be a non-negative Borel measure on $\mathbb{T}$ such that its Fourier coefficients $\widehat{\mu}(n)=\int z^n\mu(dz)$ $(n\in\mathbb{Z})$ vanish at infinity, that is $\widehat{\mu}(n)\to 0$ as $|n|\to\infty$. Then the projective orbit $O_{\rm pr}(M,f)$ is weakly sequentially closed for any $f\in L_2(\mu)$, where the multiplication operator $Mf(z)=zf(z)$ acts on $L_2(\mu)$. In particular, M is not weakly sequentially supercyclic.*

We provide an affirmative answer to Question 2 and consequently to Question 1.

THEOREM **1.2.** *There exists a Borel probability measure μ on $\mathbb{T}$ such that its Fourier coefficients vanish at infinity and the operator $Mf(z)=zf(z)$ acting on $L_2(\mu)$ is weakly supercyclic.*

Proposition 1.1 and Theorem 1.2 immediately imply the following corollary.

COROLLARY **1.3.** *There exists a weakly supercyclic unitary operator on a separable Hilbert space, which is not weakly sequentially supercyclic.*

The proof of Theorem 1.2 requires a construction of a rather complicated singular continuous measure. Curiously enough, it is much easier to give an affirmative answer to Question 1 for Banach space operators.

Given a bounded sequence $\{w_n\}_{n\in\mathbb{Z}}$ in $\mathbb{C}\setminus\{0\}$, the *weighted bilateral shift* T acting on $\ell_p(\mathbb{Z})$, $1\leqslant p<\infty$ or $c_0(\mathbb{Z})$ is defined on the canonical basis $\{e_n\}_{n\in\mathbb{Z}}$ by $Te_n=w_ne_{n-1}$. We denote

$$\beta(k,n)=\prod_{j=k}^{n}|w_j|,\qquad \text{for}\ \ k,n\in\mathbb{Z}\ \ \text{with}\ \ k\leqslant n. \tag{1}$$

In the particular case $w_n\equiv 1$ we have the *unweighted bilateral shift*, which we denote as B.

Salas [25, 24] has characterized hypercyclic and supercyclic bilateral weighted shifts in terms of weight sequences. We formulate his results in a slightly different form, however obviously equivalent to the original ones.

THEOREM **S.** *Let T be a bilateral weighted shift acting on $\ell_p(\mathbb{Z})$ with $1\leqslant p<\infty$ or $c_0(\mathbb{Z})$. Then T is hypercyclic if and only if for any $k\in\mathbb{N}_0$,*

$$\varliminf_{n\to\infty}\max\left\{\max_{|j|\leqslant k}\beta(j-n,j),\Bigl(\min_{|j|\leqslant k}\beta(j,j+n)\Bigr)^{-1}\right\}=0 \tag{2}$$

and T is supercyclic if and only if for any $k\in\mathbb{N}_0$,

$$\varliminf_{n\to+\infty}\Bigl(\max_{|j|\leqslant k}\beta(j-n,j)\Bigr)\Bigl(\min_{|j|\leqslant k}\beta(j,j+n)\Bigr)^{-1}=0. \tag{3}$$

This theorem implies, in particular, that hypercyclicity and supercyclicity of a bilateral weighted shift acting on $\ell_p(\mathbb{Z})$ with $1\leqslant p<\infty$ do not depend on p. It will be clear from the results below that it is not the case for weak hypercyclicity and weak supercyclicity.

The main result of the paper [7] by Bes, Chan and Sanders is the following.

THEOREM **BCS.** *Let T be a bilateral weighted shift acting on $\ell_p(\mathbb{Z})$, $1\leqslant p<\infty$. If T is weakly 1-sequentially hypercyclic then T is hypercyclic. If T is weakly 1-sequentially supercyclic then T is supercyclic.*

We prove the following slightly stronger statement.

PROPOSITION **1.4.** *Let T be a bilateral weighted shift acting on $\ell_p(\mathbb{Z})$, $1\leqslant p<\infty$ or $c_0(\mathbb{Z})$. If T is weakly sequentially hypercyclic then T is hypercyclic. If T is weakly sequentially supercyclic then T is supercyclic.*

In [27] it is proved that the unweighted bilateral shift B acting on $c_0(\mathbb{Z})$ is weakly supercyclic. This result is a corollary of the following stronger one.

Theorem **1.5.** *The unweighted bilateral shift B on $\ell_p(\mathbb{Z})$ is weakly supercyclic if and only if $p > 2$.*

Thus, B acting on $\ell_p(\mathbb{Z})$ for $2 < p < \infty$ provides an example of a weakly supercyclic not weakly sequentially supercyclic isometric linear operator acting on a uniformly convex Banach space. Since any $\ell_p(\mathbb{Z})$ is densely and continuously embedded into $c_0(\mathbb{Z})$, Theorem 1.5, via comparison principle, implies weak supercyclicity of B on $c_0(\mathbb{Z})$. It worth mentioning that the proof of the above result is completely different from the one in [27] for $c_0(\mathbb{Z})$. Theorems 1.2 and 1.5 are in strong contrast with Ansari and Bourdon's result [2] that a Banach space isometry can not be supercyclic. In [9] Chan and Sanders have shown that

Theorem **CS.** *The bilateral weighted shift T with the weight sequence $w_n = 2$ if $n \geqslant 0$, $w_n = 1$ if $n < 0$ acting on $\ell_p(\mathbb{Z})$ for $2 \leqslant p < \infty$ is weakly hypercyclic and non-hypercyclic.*

They also raised the following natural question.

Question **3**. *Does there exist a weakly hypercyclic non-hypercyclic bilateral weighted shift acting on $\ell_p(\mathbb{Z})$ for $1 \leqslant p < 2$?*

We answer this question negatively.

Theorem **1.6.** *Let T be a bilateral weighted shift acting on $\ell_p(\mathbb{Z})$. If $1 \leqslant p < 2$ and T is weakly hypercyclic then T is hypercyclic. If $1 \leqslant p \leqslant 2$ and T is weakly supercyclic then T is supercyclic.*

Theorem 1.6, Proposition 1.4 and Theorem CS immediately imply the following corollary.

Corollary **1.7.** *Let $1 \leqslant p < \infty$. Then any weakly hypercyclic bilateral weighted shift acting on $\ell_p(\mathbb{Z})$ is hypercyclic if and only if $p < 2$. Moreover any weakly supercyclic bilateral weighted shift acting on $\ell_p(\mathbb{Z})$ is supercyclic if and only if $p \leqslant 2$.*

Bes, Chan and Sanders [7] have also raised the following questions.

Question **4**. *Does there exist an invertible bilateral weighted shift T acting on $\ell_p(\mathbb{Z})$ such that T and T^{-1} are both weakly hypercyclic and T is not hypercyclic? Does there exist a weakly hypercyclic bilateral weighted shift T, acting on $\ell_p(\mathbb{Z})$ such that T is not supercyclic?*

We answer both questions affirmatively:

Proposition **1.8.** *There exists an invertible non-hypercyclic bilateral weighted shift T acting on $\ell_2(\mathbb{Z})$ such that both T and T^{-1} are weakly hypercyclic.*

Proposition **1.9.** *For any $p > 2$ there exists a weakly hypercyclic bilateral weighted shift acting on $\ell_p(\mathbb{Z})$, which is not supercyclic.*

Section 2 is devoted to some basic facts about weak limit points, their relation with p-sequences and antisupercyclicity, a concept that was introduced in [29]. Proposition 1.4 is proved in the end of Section 2. In Section 3 Theorem 1.4, Proposition 1.8 and Proposition 1.9 are proved. In Section 4 we prove Proposition 1.1 and Theorem 1.2, which is probably the most difficult result in this article. Theorem 1.6 is proved in Section 5. In Section 6 we discuss the tightness of certain results of the previous sections and pose some open questions related to this work.

2 Antisupercyclicity and weak closures

Throughout this section Λ is an infinite countable set. Recall that $\ell_\infty(\Lambda)$ is the space of complex valued or real valued bounded sequences $\{x_\alpha\}_{\alpha\in\Lambda}$ endowed with the supremum norm and $c_0(\Lambda)$ is the subspace of $\ell_\infty(\Lambda)$ consisting of sequences $\{x_\alpha\}_{\alpha\in\Lambda}$ such that $\{\alpha \in \Lambda : |x_\alpha| > \varepsilon\}$ is finite for each $\varepsilon > 0$. For $1 \leqslant p < \infty$, $\ell_p(\Lambda)$ is the space of sequences $x \in \ell_\infty(\Lambda)$ for which

$$\|x\|_p = \left(\sum_{\alpha\in\Lambda} |x_\alpha|^p\right)^{1/p} < \infty.$$

Of course, these spaces are isomorphic to the usual sequence spaces ℓ_p and c_0 indexed on $\mathbb{N}_0$. The point is that sometimes it is more convenient to specify a different index set. For each $\alpha \in \Lambda$, we denote by e_α the sequence in which all elements, except the α-th, whose value is one, vanish. It is well-known that $\{e_\alpha\}_{\alpha\in\Lambda}$ is an unconditional absolute Schauder basis in $\ell_p(\Lambda)$ for $1 \leqslant p < \infty$ and in $c_0(\Lambda)$. This basis is usually called *the canonical basis*. For $x \in \ell_p(\Lambda)$ and $y \in \ell_q(\Lambda)$ with $\frac{1}{p} + \frac{1}{q} = 1$, we denote

$$\langle x, y\rangle = \sum_{\alpha\in\Lambda} x_\alpha y_\alpha.$$

In what follows for a sequence $x = \{x_\alpha\}_{\alpha\in\Lambda}$ we shall usually write $\langle x, e_\alpha\rangle$ instead of x_α. The *support* of a sequence $x = \{x_\alpha\}_{\alpha\in\Lambda}$ is the set

$$\operatorname{supp}(x) = \{\alpha \in \Lambda : x_\alpha \neq 0\} = \{\alpha \in \Lambda : \langle x, e_\alpha\rangle \neq 0\}.$$

2.1 Antisupercyclicity

A bounded linear operator T acting on a Banach space $\mathcal{B}$ is called *antisupercyclic* if the sequence $\{T^n x/\|T^n x\|\}_{n\in\mathbb{N}_0}$ converges weakly to zero for any $x \in \mathcal{B}$ such that $T^n x \neq 0$ for each $n \in \mathbb{N}_0$. This is the case when the angle criterion of supercyclicity [20] is not satisfied in the strongest possible way. In Hilbert space antisupercyclicity means that the angles between any fixed vector y and the elements $T^n x$ of any orbit, not vanishing eventually, tend to $\pi/2$.

THEOREM **2.1.** *Let T be an antisupercyclic bounded linear operator acting on a Banach space $\mathcal{B}$. Then for any $x \in \mathcal{B}$, the projective orbit $O_{\rm pr}(T, x)$ is weakly sequentially closed in $\mathcal{B}$. In particular, antisupercyclic operators are never weakly sequentially supercyclic if* $\dim \mathcal{B} > 1$.

Proof. Let $x \in \mathcal{B}$ and $\{y_n\}$ be a weakly convergent sequence of elements of $O_{\rm pr}(T, x)$. For any $m \in \mathbb{N}_0$, let $L_m = \{\lambda T^n x : \lambda \in \mathbb{C},\ 0 \leqslant n \leqslant m\}$. If each y_n belongs to L_m for some m, then taking into account that L_m is weakly closed, we see that the weak limit of the sequence y_n belongs to $L_m \subset O_{\rm pr}(T, x)$. Otherwise, $T^n x \neq 0$ for any $n \in \mathbb{N}_0$ and passing to a subsequence, if necessary, we can assume that $y_n = (c_n/\|T^{m_n}x\|)T^{m_n}x$, where $c_n \in \mathbb{C}$ and m_n is a strictly increasing sequence of positive integers. Since any weakly convergent sequence is bounded, we find that $\{c_n\}$ is bounded. Antisupercyclicity of T implies that $z_n = T^{m_n}x/\|T^{m_n}x\|$ tends weakly to zero. Since c_n is bounded, we conclude that $y_n = c_n z_n$ tends weakly to zero, which is in $O_{\rm pr}(T, x)$. Hence $O_{\rm pr}(T, x)$ is weakly sequentially closed. □

2.2 Weak density and p-sequences

LEMMA **2.2.** *Let $1 < p \leqslant \infty$, $\{c_\alpha\}_{\alpha\in\Lambda}$ be a sequence of complex numbers and $\mathcal{B}_p = \ell_p(\Lambda)$ if $1 < p < \infty$, $\mathcal{B}_\infty = c_0(\Lambda)$. Then zero is in the weak closure of the set $Y = \{c_\alpha e_\alpha : \alpha \in \Lambda\}$ in the Banach space $\mathcal{B}_p$ if and only if*

$$\sum_{\alpha\in\Lambda} |c_\alpha|^{-q} = \infty, \quad \textit{where} \ \ \tfrac{1}{q} + \tfrac{1}{p} = 1. \tag{4}$$

Proof. Without loss of generality, we may assume that $c_\alpha \neq 0$ for each $\alpha \in \Lambda$, otherwise the result is trivial. Assume that (4) is not satisfied. Then $b \in \ell_q(\Lambda) = \mathcal{B}_p^*$, where $b_\alpha = |c_\alpha|^{-1}$, $\alpha \in \Lambda$. Clearly $|\langle c_\alpha e_\alpha, b\rangle| = 1$ for any $\alpha \in \Lambda$. Therefore zero is not in the weak closure of Y.

Conversely assume that (4) is satisfied. Then $b \notin \ell_q(\Lambda)$. Let $x_1, \dots, x_m \in \ell_q(\Lambda) = \mathcal{B}_p^*$ and $a_\alpha = \max\limits_{1\leqslant j\leqslant m} |\langle x_j, e_\alpha\rangle|$. Since $a = \{a_\alpha\} \in \ell_q(\Lambda)$, $b \notin \ell_q(\Lambda)$ and the entries of a and b are non-

negative, we have $\inf_{\alpha\in\Lambda} a_\alpha b_\alpha^{-1} = 0$. Finally observe that $|\langle c_\alpha e_\alpha, x_j\rangle| \leqslant a_\alpha b_\alpha^{-1}$ for any $\alpha \in \Lambda$ and $1 \leqslant j \leqslant m$. Hence $\inf_{\alpha\in\Lambda} \max_{1\leqslant j\leqslant m} |\langle c_\alpha e_\alpha, x_j\rangle| = 0$. Thus, zero is in the weak closure of Y. $\square$

Let $1 \leqslant p \leqslant \infty$. A sequence $\{x_\alpha\}_{\alpha\in\Lambda}$ of elements of a Banach space $\mathcal{B}$ is called a *p-sequence* if there exists $c > 0$ such that

$$\left\| \sum_{j=1}^{n} a_j x_{\alpha_j} \right\| \leqslant c\|a\|_p \text{ for any } n \in \mathbb{N}, \text{ any } a \in \mathbb{C}^n \text{ and any pairwise different } \alpha_1, \dots, \alpha_n \in \Lambda. \quad (5)$$

For instance, each bounded sequence in ℓ_p with disjoint supports is a p-sequence. Clearly $\{x_\alpha\}$ is a *p-sequence* if and only if there exists a bounded linear operator $S : \mathcal{B}_p \to \mathcal{B}$ such that $Se_\alpha = x_\alpha$ for each $\alpha \in \Lambda$, where $\mathcal{B}_p = \ell_p(\Lambda)$ if $1 \leqslant p < \infty$, $\mathcal{B}_\infty = c_0(\Lambda)$. The concept of p-sequence provides an easy sufficient condition for zero to belong to the weak closure of certain sequences.

LEMMA **2.3.** *Let $1 < p \leqslant \infty$ and $\{x_\alpha\}_{\alpha\in\Lambda}$ be a p-sequence in a Banach space $\mathcal{B}$ and $\{c_\alpha\}_{\alpha\in\Lambda}$ be a sequence of complex numbers, satisfying* (4). *Then zero is in the weak closure of $Y = \{c_\alpha x_\alpha : \alpha \in \Lambda\}$ in $\mathcal{B}$.*

Proof. Let $\mathcal{B}_p = \ell_p(\Lambda)$ if $1 < p < \infty$ and $\mathcal{B}_\infty = c_0(\Lambda)$. Since $\{x_\alpha\}$ is a p-sequence, there exists a bounded linear operator $S : \mathcal{B}_p \to \mathcal{B}$ such that $Se_\alpha = x_\alpha$ for each $\alpha \in \Lambda$. By Lemma 2.2, zero is in the weak closure of the set $N = \{c_\alpha e_\alpha : \alpha \in \Lambda\}$ in $\mathcal{B}_p$. Since $S(N) = Y$ and $S : \mathcal{B}_p \to \mathcal{B}$ is weak-to-weak continuous, we see that zero is in the weak closure of Y. $\square$

The previous lemma allows us to prove the following proposition, which provides sufficient conditions for weak supercyclicity and hypercyclicity.

PROPOSITION **2.4.** *Let $\mathcal{B}$ be a Banach space, $T : \mathcal{B} \to \mathcal{B}$ be a bounded linear operator, S be a subset of $\mathcal{B}$ such that $\Omega = \{\lambda x : \lambda \in \mathbb{C},\ x \in S\}$ is weakly dense in $\mathcal{B}$ and $u \in \mathcal{B}$. Assume also that for any $x \in S$, there exist $p_x \in (1, \infty]$, an infinite set $A_x \subset \mathbb{N}_0$ and maps $\alpha_x, \beta_x : A_x \to \mathbb{C}$ and $\gamma_x : A_x \to \mathbb{N}$ satisfying*

(C1) $\{\beta_x(k)T^{\gamma_x(k)}u - \alpha_x(k)x\}_{k\in A_x}$ *is a p_x-sequence in $\mathcal{B}$;*
(C2) $\sum_{k\in A_x} |\alpha_x(k)|^{q_x} = \infty$, *where $\frac{1}{p_x} + \frac{1}{q_x} = 1$.*

Then u is a weakly supercyclic vector for T.

If additionally S itself is weakly dense in $\mathcal{B}$ and $\alpha_x = \beta_x$ for each $x \in S$, then u is a weakly hypercyclic vector for T.

Proof. Let $x \in S$. Lemma 2.3 along with (C1) and (C2) implies that zero is in the weak closure of $\{\frac{\beta_x(k)}{\alpha_x(k)}T^{\gamma_x(k)}u - x : k \in A_x\}$. Thus, x is in the weak closure of $\{\frac{\beta_x(k)}{\alpha_x(k)}T^{\gamma_x(k)}u : k \in A_x\}$, which is contained in $O_{\rm pr}(T, u)$. Since x is an arbitrary element of S and $O_{\rm pr}(T, u)$ is stable under the multiplication by scalars, we see that Ω is contained in the weak closure of $O_{\rm pr}(T, x)$. Taking into account that Ω is weakly dense in $\mathcal{B}$, we observe that $O_{\rm pr}(T, u)$ is weakly dense in $\mathcal{B}$. Thus, u is a weakly supercyclic vector for T. Suppose now that S is weakly dense in $\mathcal{B}$ and $\alpha_x = \beta_x$ for each $x \in S$. Then any $x \in S$ is in the weak closure of $\{T^{\gamma_x(k)}u : k \in A_x\} \subseteq O(T, u)$. Therefore $O(T, u)$ is weakly dense in $\mathcal{B}$. Thus, u is a weakly hypercyclic vector for T. $\square$

The following lemma deals with perturbations of p-sequences.

LEMMA **2.5.** *Let $\{x_\alpha\}_{\alpha\in\Lambda}$ and $\{y_\alpha\}_{\alpha\in\Lambda}$ be two sequences in a Banach space $\mathcal{B}$, where the first one is a p-sequence for $1 \leqslant p \leqslant \infty$ and $b \in \ell_q(\Lambda)$, where $b_\alpha = \|x_\alpha - y_\alpha\|$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $\{y_\alpha\}_{\alpha\in\Lambda}$ is a p-sequence.*

Proof. Since $\{x_\alpha\}_{\alpha\in\Lambda}$ is a p-sequence, there exists $c > 0$ such that (5) is satisfied. Let $n \in \mathbb{N}$, $a \in \mathbb{C}^n$ and $\alpha_1, \dots, \alpha_n$ be pairwise different elements of Λ. Upon applying the Hölder inequality,

we obtain

$$\left\|\sum_{j=1}^n a_jy_{\alpha_j}\right\| \leqslant \left\|\sum_{j=1}^n a_jx_{\alpha_j}\right\| + \left\|\sum_{j=1}^n a_j(x_{\alpha_j}-y_{\alpha_j})\right\| \leqslant c\|a\|_p + \sum_{j=1}^n |a_j|b_{\alpha_j} \leqslant (c+\|b\|_q)\|a\|_p.$$

Hence $\{y_\alpha\}_{\alpha\in\Lambda}$ is a p-sequence. □

We end this section with a sufficient condition for being a 2-sequence in a Hilbert space.

LEMMA **2.6.** *Let $\{g_n\}_{n\in\mathbb{N}}$ be a bounded sequence in a Hilbert space $\mathcal{H}$ such that*

$$c = \sum_{1\leqslant m<n<\infty} |\langle g_n, g_m\rangle|^2 < \infty.$$

Then $\{g_n\}_{n\in\mathbb{N}}$ is a 2-sequence in $\mathcal{H}$.

Proof. Denote $d = \sup\limits_{n\in\mathbb{N}} \|g_n\|$ and let $n\in\mathbb{N}$, $a\in\mathbb{C}^n$ and $m_1,\dots,m_n$ be pairwise different positive integers. Applying the Cauchy–Schwartz inequality, we obtain

$$\left\|\sum_{j=1}^n a_jg_{m_j}\right\|^2 = \left\langle \sum_{j=1}^n a_jg_{m_j}, \sum_{k=1}^n a_kg_{m_k}\right\rangle = \sum_{j=1}^n\sum_{k=1}^n a_j\overline{a_k}\langle g_{m_j}, g_{m_k}\rangle \leqslant \sum_{j=1}^n |a_j|^2\|g_{m_j}\|^2 +$$

$$+\sum_{\substack{1\leqslant j,k\leqslant n\\ j\neq k}} |a_ja_k\langle g_{m_j}, g_{m_k}\rangle| \leqslant d^2\|a\|_2^2 + \Bigg(\sum_{\substack{1\leqslant j,k\leqslant n\\ j\neq k}} |a_ja_k|^2\Bigg)^{1/2}\Bigg(\sum_{\substack{1\leqslant j,k\leqslant n\\ j\neq k}} |\langle g_{m_j}, g_{m_k}\rangle|^2\Bigg)^{1/2} \leqslant$$

$$\leqslant d^2\|a\|_2^2 + \Bigg(\sum_{1\leqslant j,k\leqslant n} |a_ja_k|^2\Bigg)^{1/2}\Bigg(2\sum_{\substack{j,k\in\mathbb{N}\\ j>k}} |\langle g_j, g_k\rangle|^2\Bigg)^{1/2} = (d^2+(2c)^{1/2})\|a\|_2^2.$$

Hence (5) for $\Lambda=\mathbb{N}$, $x_n=g_n$ and $p=2$ is satisfied with the constant $(d^2+\sqrt{2c})^{1/2}$. Thus, $\{g_n\}_{n\in\mathbb{N}}$ is a 2-sequence. □

2.3 Proof of Proposition 1.4

In [29] it is proven that

THEOREM **A.** *A weighted bilateral shift T acting on $\ell_p(\mathbb{Z})$, $1<p<\infty$ is antisupercyclic if and only if it is not supercyclic.*

The same result is true and the same proof works when T is acting on $c_0(\mathbb{Z})$. One has to take into account that $c_0(\Lambda)$ shares the following property with $\ell_p(\Lambda)$ for $1<p<\infty$: a sequence is weakly convergent if and only if it is norm bounded and coordinatewise convergent. This fails for sequences in $\ell_1(\Lambda)$ and so does the above theorem.

Let $\mathcal{B}_p=\ell_p(\mathbb{Z})$ if $1<p<\infty$ and $\mathcal{B}_\infty=c_0(\mathbb{Z})$. The above result along with Proposition 2.1 and the comparison principle implies that a weighted bilateral shift T acting on $\mathcal{B}_p$ for $1\leqslant p\leqslant\infty$ is weakly sequentially supercyclic if and only if it is supercyclic. Moreover the projective orbits of T are weakly sequentially closed if T is not supercyclic. It remains to show that a weakly sequentially hypercyclic bilateral weighted shift is hypercyclic. The situation with hypercyclicity differs from that with supercyclicity since, as it was mentioned in [7], orbits of a non-hypercyclic weighted bilateral shift may be not weakly sequentially closed. The proof goes along the same lines as in [7], but we have to overcome few additional difficulties.

Let $1\leqslant p\leqslant\infty$ and T be a weakly sequentially hypercyclic bilateral weighted shift on $\mathcal{B}_p$. Denote by Ω_0 the set of weakly hypercyclic vectors for T. Let Ω be the set of $z\in\mathcal{B}_p$ for which

there exist $x \in \Omega$ and a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}_0}$ of positive integers such that the sequence $T^{n_k}x$ is weakly convergent to z. Since T is weakly sequentially hypercyclic, Ω is weakly sequentially dense in $\mathcal{B}_p$.

LEMMA **2.7.** *For any $z \in \Omega$, any $l \in \mathbb{N}$, any $\varepsilon > 0$ and any $y \in \mathcal{B}_p$ with finite support, there exists $v \in \mathcal{B}_p$ with finite support and $n \in \mathbb{N}$ such that $n > l$, $\|v\|_p < \varepsilon$, $\|T^n y\|_p < \varepsilon$ and $\|T^n v - z\|_p < \varepsilon$.*

Proof. Since $z \in \Omega$, there exist a weakly hypercyclic vector x for T and a strictly increasing sequence n_k of positive integers such that $T^{n_k}x$ converges weakly to z as $k \to \infty$. Since any weakly convergent sequence is bounded, there exists $M > 0$ such that $\|T^{n_k}x\|_p \leqslant M$ for any $k \in \mathbb{N}_0$. Clearly $\|u - P_{0,d}u\|_p \to 0$ as $d \to \infty$ for any $u \in \mathcal{B}_p$, where

$$P_{a,d} : \mathcal{B}_p \to \mathcal{B}_p, \qquad P_{a,d}u = \sum_{n=a-d}^{a+d} \langle u, e_n\rangle e_n.$$

Pick $r \in \mathbb{N}$ such that $\|z - P_{0,r}z\|_p < \varepsilon/2$. Since x is a weakly hypercyclic vector for T and y has finite support, there exists $m \in \mathbb{N}_0$ such that $|\langle T^m x, e_j\rangle| > M|\langle y, e_j\rangle|/\varepsilon$, whenever $j \in \operatorname{supp}(y)$. Taking into account that T is a weighted shift, we see that for any $l \in \mathbb{N}_0$,

$$|\langle T^{m+l}x, e_j\rangle| > M|\langle T^l y, e_j\rangle|/\varepsilon, \quad \text{whenever} \quad \langle T^l y, e_j\rangle \neq 0.$$

Hence $\|T^l y\|_p < \frac{\varepsilon}{M}\|T^{m+l}x\|_p$ for each $l \in \mathbb{N}_0$. In particular, for $l = n_k - m$, we have

$$\|T^{n_k-m}y\|_p < \frac{\varepsilon}{M}\|T^{n_k}x\|_p \leqslant \varepsilon, \quad \text{whenever} \quad n_k \geqslant m.$$

Denote $v_k = T^m P_{n_k,r}x$. Clearly $\|v_k\|_p \to 0$ as $k \to \infty$. Since $T^{n_k-m}v_k = P_{0,r}T^{n_k}x$ and $T^{n_k}x$ converges weakly to z, we see that $T^{n_k-m}v_k$ converges weakly to $P_{0,r}z$. Since all the vectors $T^{n_k-m}v_k$ belong to the finite dimensional range of $P_{0,r}$, we have $\|T^{n_k-m}v_k - P_{0,r}z\|_p \to 0$. Choosing k large enough, we can ensure that $n_k - m > l$, $\|v_k\|_p < \varepsilon$ and $\|T^{n_k-m}v_k - P_{0,r}z\|_p < \varepsilon/2$. Since $\|z - P_{0,r}z\|_p < \varepsilon/2$, we have $\|T^{n_k-m}v_k - z\|_p < \varepsilon$. Thus, $v = v_k$ and $n = n_k - m$ satisfy all desired conditions. $\square$

LEMMA **2.8.** *For any sequence $\{z_k\}_{k\in\mathbb{N}_0}$ of elements of Ω there exist a weakly hypercyclic for T vector $x \in \mathcal{B}_p$ and a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}_0}$ of positive integers such that $\|T^{n_k}x - z_k\|_p \to 0$ as $k \to \infty$.*

Proof. Since any sequence of elements of Ω is a subsequence of a sequence of elements of Ω, which is norm-dense in Ω, we can, without loss of generality assume that $\{z_n : n \in \mathbb{N}_0\}$ is norm-dense in Ω.

By Lemma 2.7 we can construct inductively a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}_0}$ of positive integers and $\{x_k\}_{k\in\mathbb{N}_0}$ of vectors in $\mathcal{B}_p$ with finite supports such that

$$\begin{aligned}
&\|x_k\|_p < s_k, \quad \|T^{n_k}u_k\|_p < s_k \quad \text{and} \quad \|T^{n_k}x_k - z_k\|_p < s_k, \quad \text{where} \\
&u_0 = 0, \quad u_k = x_0 + \ldots + x_{k-1} \quad \text{if} \quad k \geqslant 1, \quad s_0 = 1 \quad \text{and} \\
&s_k = 2^{-k}\min\{1, \|T^{n_0}\|_p^{-1}, \ldots, \|T^{n_{k-1}}\|_p^{-1}\} \quad \text{if} \quad k \geqslant 1.
\end{aligned}$$

Since $\|x_k\|_p \leqslant 2^{-k}$ the series $\sum\limits_{k=0}^{\infty} x_k$ is absolutely convergent in $\ell_p(\mathbb{Z})$. Let $x = \sum\limits_{k=0}^{\infty} x_k$. Then

$$\begin{aligned}
\|T^{n_k}x - z_k\|_p &= \left\|(T^{n_k}x_k - z_k) + T^{n_k}u_k + \sum_{j=k+1}^{\infty} T^{n_k}x_j\right\|_p \leqslant \\
&\leqslant 2s_k + \sum_{j=k+1}^{\infty} \|T^{n_k}\|_p s_j \leqslant 2^{-k+1} + \sum_{j=k+1}^{\infty} 2^{-j} = 3\cdot 2^{-k} \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Since $\{z_n : n \in \mathbb{N}_0\}$ is norm-dense in Ω, Ω is weakly dense in $\mathcal{B}_p$ and $\|T^{n_k}x - z_k\|_p \to 0$, we see that x is a weakly hypercyclic vector for T. Thus, x and $\{n_k\}$ satisfy all desired conditions. □

Let $\{u_n\}_{n\in\mathbb{N}_0}$ be a sequence of elements of Ω weakly convergent to $u \in \mathcal{B}_p$. By Lemma 2.8 there exist a weakly hypercyclic for T vector $x \in \mathcal{B}_p$ and a strictly increasing sequence $\{k_n\}_{n\in\mathbb{N}_0}$ of positive integers such that $\|T^{k_n}x - u_n\|_p \to 0$ as $n \to \infty$. Since u_n tends weakly to u, we have that $T^{k_n}x$ tends weakly to u. Hence $u \in \Omega$ and therefore Ω is weakly sequentially closed. Since Ω is weakly sequentially dense in $\mathcal{B}_p$, we have $\Omega = \mathcal{B}_p$. Taking a norm dense sequence $\{f_n\}_{n\in\mathbb{N}_0}$ in $\Omega = \mathcal{B}_p$ and applying Lemma 2.8 once again, we obtain $y \in \mathcal{B}_p$ and a strictly increasing sequence m_n of positive integers such that $\|T^{m_n}y - f_n\|_p \to 0$. It follows that $O(T, y)$ is norm dense in $\mathcal{B}_p$. Hence y is a hypercyclic vector for T. The proof is complete.

3 Weakly supercyclic and hypercyclic bilateral shifts

Before proving Theorem 1.5, and Propositions 1.9 and 1.10, we, using Proposition 2.4, shall derive sufficient conditions for weak hypercyclicity and weak supercyclicity of invertible weighted shifts in terms of weight sequences. Our sufficient condition of weak hypercyclicity of a bilateral weighted shift differs from the one of Chan and Sanders [9] and is fairly easier to handle. In fact it is possible, using basically the same proof, to generalize our criteria for non-invertible bilateral weighted shifts, but the conditions become too heavy in this case.

Recall that the *density* of a subset $A \subset \mathbb{N}_0$ is the limit $\lim\limits_{n\to\infty} \frac{N(n)}{n}$, where N is the counting function of A, that is, $N(n)$ is the number of elements of the set $\{m \in A : m \leqslant n\}$. The following elementary lemma can be found in many places, see for instance [19], Chapter 1.

LEMMA **3.1.** *Let A be a subset of $\mathbb{N}_0$ of positive density and $\{s_n\}_{n\in\mathbb{N}_0}$ be a monotonic sequence of positive numbers. Then $\sum\limits_{n\in\mathbb{N}_0} s_n = \infty$ if and only if $\sum\limits_{n\in A} s_n = \infty$.*

For a sequence $x = \{x_n\}_{n\in\mathbb{Z}}$ of complex numbers denote

$$\gamma(x) = \max_{n\in \mathrm{supp}\,(x)} |n|.$$

In the following two lemmas $\mathcal{B}_p = \ell_p(\mathbb{Z})$ if $1 \leqslant p < \infty$ and $\mathcal{B}_\infty = c_0(\mathbb{Z})$.

LEMMA **3.2.** *Let $1 \leqslant p < \infty$, and $\{a_n\}_{n\in\mathbb{N}_0}$ be a sequence of non-negative numbers such that $\lim\limits_{n\to\infty} a_n = \infty$. Then there exists a map $\varkappa : \mathbb{N}_0 \to \mathcal{B}_p$ such that*

(U1) *the set $S = \varkappa(\mathbb{N}_0)$ consists of vectors with finite support and is norm-dense in $\mathcal{B}_p$;*

(U2) *$\gamma(\varkappa(n)) \leqslant a_n$ and $\|\varkappa(n)\|_p \leqslant a_n$ for each $n \in \mathbb{N}_0$;*

(U3) *for each $x \in S \setminus \{0\}$ the set $\varkappa^{-1}(x)$ has positive density.*

Proof. Take a dense in $\mathcal{B}_p$ sequence $\{x_n\}_{n\in\mathbb{N}_0}$ of pairwise different non-zero vectors with finite support. Since $\lim\limits_{n\to\infty} a_n = \infty$, we can pick a strictly increasing sequence $\{m_n\}_{n\in\mathbb{N}_0}$ of positive integers such that $\|x_n\|_p \leqslant a_k$ and $\gamma(x_n) \leqslant a_k$ for each $k \geqslant m_n$. Choose a strictly increasing sequence $\{p_n\}_{n\in\mathbb{N}_0}$ of prime numbers such that $p_n > m_n$ for any $n \in \mathbb{N}_0$ and put

$$A_0 = \{jp_0 + 1 : j \in \mathbb{N}\} \ \text{ and } \ A_n = \{p_0 \cdot \ldots \cdot p_{n-1}(jp_n + 1) : j \in \mathbb{N}\} \ \text{ for } \ n \geqslant 1.$$

Each A_n is an arithmetic progression and therefore has positive density. From easy divisibility considerations it follows that the sets A_n are disjoint. This allows us to define the map kappa by setting $\varkappa(m) = x_n$ if $m \in A_n$ and $\varkappa(m) = 0$ if $m \in \mathbb{N}_0 \setminus \bigcup\limits_{n=0}^{\infty} A_n$.

Since $S = \varkappa(\mathbb{N}_0) = \{0\} \cup \{x_n : n \in \mathbb{N}_0\}$, we see that S is dense in $\ell_p(\mathbb{Z})$ and consists of vectors with finite support. Since $\varkappa^{-1}(x_n) = A_n$, condition (U3) is satisfied. Let $m \in A_n$. From the definition of A_n it follows that $m \geqslant m_n$ and therefore $\varkappa(m) = x_n$ satisfies (U2). If $m \in \mathbb{N}_0 \setminus \bigcup\limits_{n=0}^{\infty} A_n$, then $\varkappa(m) = 0$ and (U2) is trivially satisfied. Thus, $\varkappa$ satisfies all required conditions. $\square$

For $n \in \mathbb{N}_0$ and $m \in \mathbb{Z}$ denote

$$L(m,n) = \{k \in \mathbb{Z} : |k-m| \leqslant n\}. \tag{6}$$

Clearly

$$L(a,b) \cap L(c,d) = \varnothing \text{ if and only if } |a-c| > b+d. \tag{7}$$

Lemma **3.3.** *Let T be a bilateral weighted shift acting on $\mathcal{B}_p$, $1 \leqslant p \leqslant \infty$, $\{a_n\}_{n\in\mathbb{N}_0}$, $\{r_n\}_{n\in\mathbb{N}_0}$ be monotonically non-decreasing sequences of non-negative integers such that such that $r_n - r_{n-1} - r_{n-2} > a_n + a_{n-1}$ for any $n \geqslant 2$, $\{x_{n,k}\}_{n,k\in\mathbb{N}_0}$ be a double sequence of vectors from $\ell_p(\mathbb{Z})$ such that $\gamma(x_{n,k}) \leqslant a_n$ for each $n,k \in \mathbb{N}_0$ and*

$$y_k = \sum_{\substack{n\in\mathbb{N}_0,\ n\neq k\\ a_n<(r_k-r_{k-1})/2}} T^{r_k-r_n} x_{n,k} \in \mathcal{B}_p, \quad k \in \mathbb{N}.$$

Then y_k have disjoint supports.

Proof. Since $\gamma(x_{n,k}) \leqslant a_n$, we have $\operatorname{supp}(x_{n,k}) \subseteq L(0,a_n)$ for each $n,k \in \mathbb{N}_0$, where the sets $L(m,n)$ are defined in (6). Therefore $\operatorname{supp}(T^{r_k-r_n}x_{n,k}) \subseteq L(r_n - r_k, a_n)$ for each $n,k \in \mathbb{N}_0$. Hence,

$$\operatorname{supp}(y_k) \subseteq \bigcup_{\substack{n\in\mathbb{N}_0,\ n\neq k\\ a_n<(r_k-r_{k-1})/2}} L(r_n - r_k, a_n).$$

Let $k,l \in \mathbb{N}$ and $k > l$. We have to show that $\operatorname{supp}(y_k) \cap \operatorname{supp}(y_l) = \varnothing$. According to the last display and (7) it suffices to verify that

$$\begin{array}{l} |(r_n - r_k) - (r_m - r_l)| > a_n + a_m \\ \text{if } m,n \in \mathbb{N}_0,\ a_n < (r_k - r_{k-1})/2,\ a_m < (r_l - r_{l-1})/2,\ n \neq k \text{ and } m \neq l. \end{array} \tag{8}$$

Let $m,n \in \mathbb{N}_0$ be such that $a_n < (r_k - r_{k-1})/2$, $a_m < (r_l - r_{l-1})/2$, $n \neq k$ and $m \neq l$.

Case $m \neq n$. Denote $j = \max\{n,k,m,l\}$. Since $n \neq k$, $m \neq l$, $k \neq l$ and $m \neq n$, we see that $j \geqslant 2$ and no cancelation occurs in the expression $(r_n - r_k) - (r_m - r_l)$. Thus,

$$|(r_n - r_k) - (r_m - r_l)| \geqslant r_j - r_{j-1} - r_{j-2} > a_j + a_{j-1} \geqslant a_k + a_l.$$

Case $m = n$. Since $k > l$, we have

$$|(r_n - r_k) - (r_m - r_l)| = r_k - r_l \geqslant r_k - r_{k-1} > 2a_n = a_n + a_m.$$

Thus, (8) is satisfied and therefore the supports of y_k are disjoint. $\square$

Let T be an invertible bilateral weighted shift acting on $\ell_p(\mathbb{Z})$, $1 < p < \infty$ and $w = \{w_n\}_{n\in\mathbb{Z}}$ be its weight sequence. As usual $\beta(a,b)$ stand for the numbers defined in (1).

Proposition **3.4.** *Suppose that there exist a sequence $\{r_n\}_{n\in\mathbb{N}_0}$ of positive integers and sequences $\{\alpha_n\}_{n\in\mathbb{N}_0}$, $\{\rho_n\}_{n\in\mathbb{N}_0}$ of positive numbers such that*

(W1) $\alpha_n \to \infty$ *as* $n \to \infty$;

(W2) $r_{n+2} - r_{n+1} - r_n \to \infty$ *as* $n \to \infty$;

(W3) $\sum\limits_{n=0}^{\infty} \rho_n^p \alpha_n^p \beta(1, r_n)^{-p} < \infty$;

(W4) $\sum\limits_{k=1}^{\infty} \left(\max\limits_{1 \leqslant m \leqslant k} \left(\sum\limits_{n=0}^{m-1} \frac{\alpha_n^p \rho_n^p}{\rho_m^p} \beta(r_n - r_m + 1, 0)^p + \sum\limits_{n=m+1}^{\infty} \frac{\alpha_n^p \rho_n^p}{\rho_m^p} \beta(1, r_n - r_m)^{-p} \right) \right)^{-\frac{1}{p-1}} = \infty.$

Then T is weakly supercyclic.

If a sequence $\{r_n\}_{n\in\mathbb{N}_0}$ of positive integers and a sequence $\{\alpha_n\}_{n\in\mathbb{N}_0}$ of positive numbers can be chosen such that conditions (W1–W4) *are satisfied with $\rho_n \equiv 1$, then T is weakly hypercyclic.*

Proof. Since T is invertible, there exists $c > 1$ such that $c^{-1} \leqslant |w_n| \leqslant c$ for each $n \in \mathbb{Z}$. Therefore $c^{a-b-1} \leqslant \beta(a, b) \leqslant c^{b-a+1}$ for each $a, b \in \mathbb{Z}$, $a \leqslant b$. Moreover,

$$\frac{\beta(a,b)}{\beta(a+j,b+j)} \leqslant c^{2|j|} \text{ for each } a, b, j \in \mathbb{Z},\ a \leqslant b. \tag{9}$$

Let x be a vector from $\ell_p(\mathbb{Z})$ with finite support. Using (9), we obtain that for any $n \in \mathbb{N}$,

$$\|T^{-n}x\|_p \leqslant \|x\|_p \max_{|j|\leqslant\gamma(x)} (\beta(j+1, j+n))^{-1} \leqslant \frac{\|x\|_p}{\beta(1,n)} \max_{|j|\leqslant\gamma(x)} \frac{\beta(1,n)}{\beta(j+1,j+n)} \leqslant \frac{\|x\|_p\, c^{2\gamma(x)}}{\beta(1,n)}; \tag{10}$$

$$\begin{aligned} \|T^n x\|_p &\leqslant \|x\|_p \max_{|j|\leqslant\gamma(x)} \beta(j-n+1, j) \leqslant \\ &\leqslant \|x\|_p \beta(1-n, 0) \max_{|j|\leqslant\gamma(x)} \frac{\beta(j-n+1,j)}{\beta(1-n,0)} \leqslant \|x\|_p \beta(1-n,0) c^{2\gamma(x)}. \end{aligned} \tag{11}$$

Note that the conditions (W1–W4) remain valid if we replace r_n, α_n and ρ_n by r_{n+m}, α_{n+m} and ρ_{n+m} respectively for any fixed non-negative integer m. Thus, taking (W1) into account, we can, without loss of generality, assume that $r_1 > r_0$ and $r_n > r_{n-1} + r_{n-2}$ for each $n \geqslant 2$.

According to (W3) we have

$$\sum_{n=k+1}^{\infty} \rho_n^p \alpha_n^p \beta(1, r_n - r_k)^{-p} \leqslant c^{r_k} \sum_{n=k+1}^{\infty} \rho_n^p \alpha_n^p \beta(1, r_n)^{-p} < \infty \text{ for each } k \in \mathbb{N}_0.$$

Hence we can pick a strictly increasing sequence $\{m_k\}_{k\in\mathbb{N}_0}$ of positive integers such that

$$\sum_{n=m_k}^{\infty} \rho_n^p \alpha_n^p \beta(1, r_n - r_k)^{-p} < \rho_k^p 2^{-pk} \text{ for each } k \in \mathbb{N}_0. \tag{12}$$

Now choose a monotonically non-decreasing sequence $\{a_n : n \in \mathbb{N}_0\}$ of non-negative integers tending to infinity slowly enough to ensure that

$$2a_{m_k} < r_k - r_{k-1} \text{ for each } k \in \mathbb{N}; \tag{13}$$

$$a_n c^{2a_n} \leqslant \alpha_n \text{ for each } n \in \mathbb{N}_0; \tag{14}$$

$$a_n + a_{n-1} < r_n - r_{n-1} - r_{n-2} \text{ for each } n \geqslant 2. \tag{15}$$

According to Lemma 3.2 there exists a map $\varkappa : \mathbb{N}_0 \to \ell_p(\mathbb{Z})$ such that the conditions (U1), (U2) and (U3) are satisfied. Since $\operatorname{supp}(T^{-r_n}\varkappa(n)) \subseteq L(r_n, a_n)$, from (15) and (7) it follows that the supports of $T^{-r_n}\varkappa(n)$ are disjoint. The estimates (10), (U2) and (14) imply that

$$\|T^{-r_n}\varkappa(n)\|_p \leqslant a_n c^{2a_n} \beta(1, r_n)^{-1} \leqslant \alpha_n \beta(1, r_n)^{-1} \text{ for each } n \in \mathbb{N}_0.$$

By (W3), $\sum\limits_{n=0}^{\infty} \rho_n^p \|T^{-r_n}\varkappa(n)\|_p^p < \infty$. Since the supports of $T^{-r_n}\varkappa(n)$ are disjoint, the series

$$u = \sum_{n=0}^{\infty} \rho_n T^{-r_n}\varkappa(n)$$

is norm-convergent in $\ell_p(\mathbb{Z})$. It suffices to prove that u is a weakly supercyclic vector for T and that u is a weakly hypercyclic vector for T if $\rho_n \equiv 1$.

Clearly $T^{r_k}u = \rho_k\varkappa(k) + v_k + z_k + y_k$, where

$$v_k = \sum_{\substack{a_n \geqslant (r_k - r_{k-1})/2 \\ n>k}} \rho_n T^{r_k - r_n}\varkappa(n), \quad z_k = \sum_{\substack{a_n < (r_k - r_{k-1})/2 \\ n>k}} \rho_n T^{r_k - r_n}\varkappa(n), \quad y_k = \sum_{n=0}^{k-1} \rho_n T^{r_k - r_n}\varkappa(n).$$

From (11), (10), (U2) and (14), we have

$$\|T^{r_k - r_n}\varkappa(n)\|_p \leqslant a_n c^{2a_n}\beta(1, r_n - r_k)^{-1} \leqslant \alpha_n \beta(1, r_n - r_k)^{-1}, \qquad \text{if } n > k; \tag{16}$$

$$\|T^{r_k - r_n}\varkappa(n)\|_p \leqslant a_n c^{2a_n}\beta(r_n - r_k + 1, 0) \leqslant \alpha_n \beta(r_n - r_k + 1, 0), \quad \text{if } n < k. \tag{17}$$

Since T preserves disjointness of the supports and the supports of $T^{-r_n}\varkappa(n)$ are disjoint, we see that for any $k \in \mathbb{N}_0$ the supports of $T^{r_k - r_n}\varkappa(n)$, $n \in \mathbb{N}_0$ are also disjoint. Hence,

$$\|v_k\|_p^p = \sum_{\substack{a_n \geqslant (r_k - r_{k-1})/2 \\ n>k}} \rho_n^p \|T^{r_k - r_n}\varkappa(n)\|_p^p \ \text{ for any } k \in \mathbb{N}.$$

Applying (13), (12) and (16), we obtain

$$\|v_k\|_p^p \leqslant \sum_{n \geqslant m_k} \rho_n^p \|T^{r_k - r_n}\varkappa(n)\|_p^p \leqslant \sum_{n \geqslant m_k} \rho_n^p \alpha_n^p \beta(1, r_n - r_k)^{-p} \leqslant \rho_k^p 2^{-kp} \ \text{ for any } k \in \mathbb{N}. \tag{18}$$

Analogously, applying (16) and (17), we get

$$\|z_k\|_p^p \leqslant \sum_{n=k+1}^{\infty} \rho_n^p \|T^{r_k - r_n}\varkappa(n)\|_p^p \leqslant \sum_{n=k+1}^{\infty} \alpha_n^p \rho_n^p \beta(1, r_n - r_k)^{-p};$$

$$\|y_k\|_p^p = \sum_{n=0}^{k-1} \rho_n^p \|T^{r_k - r_n}\varkappa(n)\|_p^p \leqslant \sum_{n=0}^{k-1} \alpha_n^p \rho_n^p \beta(r_n - r_k + 1, 0)^p.$$

Hence

$$\|z_k + y_k\|_p^p = \|y_k\|_p^p + \|z_k\|_p^p \leqslant \rho_k^p \xi_k \leqslant \rho_k^p \theta_k, \quad \text{where}$$

$$\xi_m = \sum_{n=0}^{m-1} \frac{\alpha_n^p \rho_n^p}{\rho_m^p} \beta(r_n - r_m + 1, 0)^p + \sum_{n=m+1}^{\infty} \frac{\alpha_n^p \rho_n^p}{\rho_m^p} \beta(1, r_n - r_m)^{-p} \ \text{ and } \ \theta_k = \max_{1 \leqslant m \leqslant k} \xi_m.$$

In view of (15), Lemma 3.3 implies that the sequence $\{y_k + z_k\}_{k \in \mathbb{N}}$ has disjoint supports. Hence $\{\rho_k^{-1}\theta_k^{-1/p}(y_k + z_k)\}$ is a p-sequence in $\ell_p(\mathbb{Z})$ as a bounded sequence with disjoint supports. Since the sequence θ_k is monotonically non-decreasing, it is bounded from below by a positive constant and therefore from (18) and Lemma 2.5 on small perturbations of p-sequences it follows that $\{\rho_k^{-1}\theta_k^{-1/p}(v_k + y_k + z_k)\}$ is a p-sequence in $\ell_p(\mathbb{Z})$. Since $T^{r_k}u = \rho_k\varkappa(k) + v_k + z_k + y_k$, we see that $\{\rho_k^{-1}\theta_k^{-1/p}T^{r_k}u - \theta_k^{-1/p}\varkappa(k)\}_{k \in \mathbb{N}}$ is a p-sequence in $\ell_p(\mathbb{Z})$.

Pick $x \in S = \varkappa(\mathbb{N}_0) \setminus \{0\}$ and let $A_x = \varkappa^{-1}(x)$. By (U3) A_x has positive density. According to (W4), $\sum\limits_{k=1}^{\infty} \theta_k^{-1/(p-1)} = \infty$. By Lemma 3.1 $\sum\limits_{k \in A_x} \theta_k^{-1/(p-1)} = \infty$, or equivalently, $\sum\limits_{k \in A_x} (\theta_k^{-1/p})^q = \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$. Since $\varkappa(k) = x$ for any $k \in A_x$, we see that $\{\rho_k^{-1}\theta_k^{-1/p}T^{r_k}u - \theta_k^{-1/p}x\}_{k \in A_x}$ is a p-sequence in $\ell_p(\mathbb{Z})$. Since S is dense in $\ell_p(\mathbb{Z})$, Proposition 2.4 implies that u is a weakly supercyclic vector for T. If additionally $\rho_n \equiv 1$, Proposition 2.4 implies that u is a weakly hypercyclic vector for T. $\Box$

Remark. An analog of Proposition 3.4 holds for invertible bilateral weighted shifts on $c_0(\mathbb{Z})$ and the proof is basically the same. One has to replace conditions (W3) and (W4) by

(W3′) $\lim\limits_{n \to \infty} \rho_n \alpha_n \beta(1, r_n)^{-1} = 0$;

(W4′) $\sum\limits_{k=1}^{\infty} \left(\max\limits_{1 \leqslant m \leqslant k} \left(\max\limits_{0 \leqslant n \leqslant m-1} \frac{\alpha_n \rho_n}{\rho_m} \beta(r_n - r_m + 1, 0) + \max\limits_{n \geqslant m+1} \frac{\alpha_n \rho_n}{\rho_m} \beta(1, r_n - r_m)^{-1} \right) \right)^{-1} = \infty.$

3.1 Proof of Theorem 1.5

We have to prove that B is weakly supercyclic for $p > 2$. Take $\{r_k\}$ being any sequence of positive integers, satisfying the condition (W2) of Proposition 3.4, for instance $r_k = 2^k$. Clearly for the unweighted bilateral shift, we have $\beta(a, b) = 1$ for each $a, b \in \mathbb{Z}$, $a \leqslant b$. Therefore, if (W3) is satisfied, then the k-th term in the sum in (W4) is bounded from below by $c\rho_k^{p/(p-1)}$ for some positive constant c. Thus, all conditions of Proposition 3.4 will be satisfied if we find sequences $\{\alpha_n\}_{n \in \mathbb{N}_0}$ and $\{\rho_n\}_{n \in \mathbb{N}_0}$ of positive numbers such that $\alpha_n \to \infty$, $\sum\limits_{n=0}^{\infty} \alpha_n^p \rho_n^p < \infty$ and $\sum\limits_{k=0}^{\infty} \rho_k^{p/(p-1)} = \infty$. This can be achieved by choosing $\alpha_n = \ln(n+2)$ and $\rho_n = (n+1)^{-1/p}(\ln(n+2))^{-2}$.

3.2 Proof of Proposition 1.8

Consider the sequence $w = \{w_n\}_{n \in \mathbb{Z}}$ defined by the formula

$$w_m = \begin{cases} 2 & \text{if } 7 \cdot 9^k < m \leqslant 9^{k+1},\ k \text{ even, or } -11 \cdot 9^k \leqslant m < -9^{k+1},\ k \text{ odd;} \\ 1/2 & \text{if } 9^{k+1} \leqslant m < 11 \cdot 9^{k+1},\ k \text{ even, or } -9^{k+1} < m \leqslant -7 \cdot 9^k,\ k \text{ odd;} \\ 1 & \text{otherwise.} \end{cases}$$

In this section T stands for the bilateral weighted shift with the weight sequence w, acting on $\ell_2(\mathbb{Z})$. Obviously T is invertible. From definition of the weight sequence $\{w_n\}$ it follows that $\max\{\beta(-n, 0), (\beta(0, n))^{-1}\} \geqslant 1$ for each $n \in \mathbb{N}_0$. Hence T is not hypercyclic according to Theorem S. It remains to show that T and T^{-1} are weakly hypercyclic.

Consider the sequences $\rho_n = 1$, $r_n = 9^{2n+1}$ and $\alpha(n) = \ln\ln(n+4)$, $n \in \mathbb{N}_0$. Conditions (W1) and (W2) of Proposition 3.4 are trivially satisfied. Using the definition of the weight sequence $\{w_n\}$, one can easily verify that for even $k \in \mathbb{N}_0$

$$\beta(1, a) = 2^{a - 7 \cdot 9^k} \quad \text{and} \quad \beta(-a + 1, 0) = 1 \quad \text{if } 7 \cdot 9^k < a \leqslant 9^{k+1}. \tag{19}$$

This implies that $\beta(1, r_n - r_k) > 2^{9^{2n}}$ if $n > k$ and $\beta(r_n - r_k + 1, 0) = 1$ if $n < k$. Now it is an elementary exercise to show that (W3) and (W4) for $p = 2$ are also satisfied. Thus, T is weakly hypercyclic according to Proposition 3.4.

Since the operator T^{-1} is similar to the weighted bilateral shift $\widetilde{T}$ with the weight sequence $\widetilde{w}_n = w_{-n}^{-1}$, it suffices to verify that $\widetilde{T}$ is weakly hypercyclic. This follows from Proposition 3.4 similarly via choosing the sequences $\rho_n = 1$, $r_n = 9^{2n+2}$ and $\alpha(n) = \ln\ln(n+4)$, $n \in \mathbb{N}_0$. Indeed, (19) is satisfied for odd $k \in \mathbb{N}_0$ for the weight sequence $\widetilde{w}_n$.

3.3 Proof of Proposition 1.9

Let $p > 2$ and $\varphi : [0, \infty) \to [1, \infty)$ be the function defined as

$$\varphi(t) = (t+1)^{1/p}(\log_2(t+2))^{2/p}.$$

Consider the sequence $w = \{w_n\}_{n\in\mathbb{Z}}$ of positive numbers defined by the formula

$$w_m = \begin{cases} \left(\frac{\varphi(k+1)}{\varphi(k)}\right)^{3^{-k}/2} & \text{if } 3^n - 3^{k+1} < |m| \leqslant 3^n - 3^k,\ n, k \in \mathbb{N}_0,\ n \geqslant k+2; \\ \left(\frac{\varphi(n+1)}{\varphi(n)^2}\right)^{3^{-n}} & \text{if } 3^n < |m| \leqslant 3^{n+1} - 3^n,\ n \in \mathbb{N}_0; \\ 1 & \text{if } m = 0 \text{ or } |m| = 3^n,\ n \in \mathbb{N}_0. \end{cases}$$

In this section T stands for the bilateral weighted shift with the weight sequence w, acting on $\ell_p(\mathbb{Z})$. It suffices to prove that T is weakly hypercyclic and non-supercyclic.

It is easy to see that the sequence w is symmetric: $w_n = w_{-n}$, $n \in \mathbb{N}_0$ and that $w_n \to 1$ as $|n| \to \infty$. Hence there exists $c > 1$ such that $c^{-1} \leqslant w_n \leqslant c$ for each $n \in \mathbb{Z}$. Therefore T is invertible. Using the definition of w it is straightforward to verify that

$$\beta(1, 3^n - 3^k) = \varphi(n)/\varphi(k) \text{ if } n, k \in \mathbb{N}_0 \text{ and } n > k, \tag{20}$$

$$\beta(1, 3^n) = \varphi(n) \text{ for } n \in \mathbb{N}_0. \tag{21}$$

We shall prove that T is weakly hypercyclic. Consider the sequences $\rho_n = 1$, $r_n = 3^n$ and $\alpha(n) = \ln\ln(n+4)$, $n \in \mathbb{N}_0$. Conditions (W1) and (W2) of Proposition 3.4 are trivially satisfied. Using (21), we see that

$$\sum_{n=0}^{\infty} \rho_n^p \alpha_n^p \beta(1, r_n)^{-p} = \sum_{n=0}^{\infty} \alpha_n^p \varphi(n)^{-p} < \infty.$$

Hence (W3) is also satisfied. Let now

$$\xi_m = \sum_{n=0}^{m-1} \frac{\alpha_n^p \rho_n^p}{\rho_m^p} \beta(r_n - r_m + 1, 0)^p + \sum_{n=m+1}^{\infty} \frac{\alpha_n^p \rho_n^p}{\rho_m^p} \beta(1, r_n - r_m)^{-p} \quad \text{for } m \in \mathbb{N}.$$

Since $\rho_n \equiv 1$ and the weight sequence w is symmetric, we using (20) obtain

$$\xi_m = \sum_{n=0}^{m-1} \alpha_n^p \beta(1, r_m - r_n)^p \frac{w_0^p}{w_{r_n - r_m}^p} + \sum_{n=m+1}^{\infty} \alpha_n^p \beta(1, r_n - r_m)^{-p} \leqslant$$

$$\leqslant c^p \sum_{n=0}^{m-1} \alpha_n^p \frac{\varphi(m)^p}{\varphi(n)^p} + \sum_{n=m+1}^{\infty} \alpha_n^p \frac{\varphi(m)^p}{\varphi(n)^p} \leqslant c^p \varphi(m)^p \sum_{n=0}^{\infty} \alpha_n^p \varphi(n)^{-p} = A(p)\varphi(m)^p,$$

where $A(p)$ is a positive constant depending only on p. Since $p > 2$, we have

$$\sum_{k=1}^{\infty} \Bigl(\max_{1\leqslant m\leqslant k} \xi_m\Bigr)^{-\frac{1}{p-1}} \geqslant A(p)^{-1/(p-1)} \sum_{k=1}^{\infty} \varphi(k)^{-p/(p-1)} = \infty.$$

Thus, (W4) is also satisfied and Proposition 3.4 implies that T is weakly hypercyclic. It remains to notice that according to Theorem S, a bilateral weighted shift with symmetric weight sequence is never supercyclic. The proof is complete.

4 The multiplication operator M: proof of Theorem 1.2

Let $\mathcal{M} = \mathcal{M}(\mathbb{T})$ be the space of σ-additive complex-valued Borel measures on the unit circle $\mathbb{T}$. We denote the set of non-negative measures $\mu \in \mathcal{M}$ as $\mathcal{M}_+$. It is well-known that $\mathcal{M}$ is a Banach space with respect to the variation norm $\|\mu\| = |\mu|(\mathbb{T})$, where $|\mu| \in \mathcal{M}_+$ is the variation of μ. That is, $|\mu|(A)$ is the supremum of $\sum\limits_n |\mu(A_n)|$, where A_n are disjoint Borel subsets of A. The set of measures $\mu \in \mathcal{M}$, whose Fourier coefficients $\widehat{\mu}(n) = \int\limits_{\mathbb{T}} z^n \, d\mu(z)$, $n \in \mathbb{Z}$ tend to zero when $|n| \to \infty$ will be denoted by $\mathcal{M}_0$.

4.1 Proof of Proposition 1.1

Let $f, g \in L_2(\mu)$. Then $\langle M^n f, g\rangle = \widehat{\nu}(n)$, where $\nu \in \mathcal{M}$ is absolutely continuous with respect to μ with the density $\frac{d\nu}{d\mu}(z) = f(z)\overline{g(z)}$. Since a measure absolutely continuous with respect to a measure from $\mathcal{M}_0$ also belongs to $\mathcal{M}_0$, see [17], we find that $\nu \in \mathcal{M}_0$, that is, $\widehat{\nu}(n) \to 0$. Hence $\langle M^n f, g\rangle / \|M^n f\| = \widehat{\nu}(n)/\|f\| \to 0$ for any non-zero $f \in L_2(\mu)$ and any $g \in L_2(\mu)$. Therefore the sequence $\{M^n f/\|M^n f\|\}$ tends weakly to zero. Thus, M is antisupercyclic. It remains to apply Theorem 2.1.

4.2 Weak convergence of measures

We need to introduce further notation. For a Borel measurable set $K \subset \mathbb{T}$ we denote by $\mathcal{M}(K)$ the set of $\mu \in \mathcal{M}$ such that $|\mu|(\mathbb{T} \setminus K) = 0$. The *support* of a measure $\mu \in \mathcal{M}$ is

$$\operatorname{supp}(\mu) = \bigcap \{K \subset \mathbb{T} : K \text{ is closed and } \mu \in \mathcal{M}(K)\}.$$

Recall that the weak topology σ on $\mathcal{M}$ is the topology generated by the functionals

$$\mu \mapsto [\mu, f] = \int\limits_{\mathbb{T}} f(z)\, \mu(dz), \quad f \in \mathcal{C}(\mathbb{T}),$$

that is, σ is the weakest topology with respect to which the functionals $\mu \mapsto [\mu, f]$ are continuous. For $\mu \in \mathcal{M}$ and a Borel-measurable set $A \subset \mathbb{T}$, μ_A stands for the restriction of μ to A, that is $\mu_A \in \mathcal{M}$ is defined by $\mu_A(B) = \mu(A \cap B)$. An *interval* I of $\mathbb{T}$ is a non-empty **open** connected subset of $\mathbb{T}$ and $|I|$ will denote its length.

LEMMA **4.1.** *Let $\mathbb{I}^n = \{I_1^n, I_2^n, \dots, I_{k_n}^n\}$, $n \in \mathbb{N}$ be a family of disjoint intervals of $\mathbb{T}$ satisfying*

(L1) *for each $n \in \mathbb{N}$, any element of $\mathbb{I}^{n+1}$ is contained in some element of $\mathbb{I}^n$;*

(L2) $\max\limits_{1 \leqslant j \leqslant k_n} |I_j^n| \to 0$ *as $n \to \infty$.*

Let also $\mu \in \mathcal{M}_+$ and $\mu^n \in \mathcal{M}_+$, $n \in \mathbb{N}$ be such that

(L3) $\mu(\mathbb{T}) = \mu^n(\mathbb{T}) = \sum\limits_{j=1}^{k_n} \mu^m(I_j^n) = \sum\limits_{j=1}^{k_n} \mu(I_j^n)$ *for any $n \in \mathbb{N}$;*

(L4) $\mu(I_j^n) = \mu^n(I_j^n)$ *for any $n \in \mathbb{N}$ and $j = 1, \dots, k_n$.*

Then the $\mu^n \xrightarrow{\ \sigma\ } \mu$ as $n \to \infty$. Moreover $\mu^n_{I_j^m} \xrightarrow{\ \sigma\ } \mu_{I_j^m}$ as $n \to \infty$ for any $m \in \mathbb{N}$ and $1 \leqslant j \leqslant k_m$.

P r o o f. Let $f \in \mathcal{C}(\mathbb{T})$ and $\varepsilon > 0$. Since f is uniformly continuous, condition (L2) implies the existence of $a \in \mathbb{N}$ and $z_1, \dots, z_{k_a} \in \mathbb{C}$ for which $|f(z) - z_j| \leqslant \varepsilon$ if $z \in I_j^a$ and $1 \leqslant j \leqslant k_a$. According to (L1), (L3) and (L4), $\mu^n(I_j^a) = \mu(I_j^a)$ for each $n \geqslant a$ and $1 \leqslant j \leqslant k_a$. Hence for each $n \geqslant a$, we have

$$|[\mu_{I_j^a}, f] - c_j z_j| \leqslant c_j\varepsilon \ \text{ and } \ |[\mu^n_{I_j^a}, f] - c_j z_j| \leqslant c_j\varepsilon \ \text{ for } 1 \leq j \leqslant n_a, \text{ where } c_j = \mu(I_j^a).$$

Therefore $|[\mu_{I_j^a}, f] - [\mu^n_{I_j^a}, f]| \leqslant 2c_j\varepsilon$ if $n \geqslant a$ and $1 \leqslant j \leqslant k_a$. Thus summing over j, we obtain

$$|[\mu, f] - [\mu^n, f]| \leqslant 2\varepsilon \sum_{j=1}^{k_a} c_j = 2\mu(\mathbb{T})\varepsilon \ \text{ for } \ n \geqslant a.$$

Hence $\mu^n \xrightarrow{\sigma} \mu$ as $n \to \infty$.

Fix now $m, j \in \mathbb{N}$ such that $1 \leqslant j \leqslant k_m$. One can easily verify that conditions (L1–L4) remain valid if we replace $\mathbb{I}^n$ by $\mathbb{I}^{n+m}$, μ^n by $\mu^{n+m}_{I_j^m}$ and μ by $\mu_{I_j^m}$. From what is already proven, follows that $\mu^{n+m}_{I_j^m} \xrightarrow{\sigma} \mu_{I_j^m}$. Hence $\mu^n_{I_j^m} \xrightarrow{\sigma} \mu_{I_j^m}$. $\square$

Recall that a set $A \subset \mathbb{T}$ is called *independent* if $\prod\limits_{j=1}^{m} z_j^{n_j} \neq 1$ for each pairwise different points $z_1, \dots, z_m \in A$ and each non-zero vector $(n_1, \dots, n_m) \in \mathbb{Z}^m$.

The set of probability measures $\mu \in \mathcal{M}$, will be denoted by $\mathcal{P}$, $\mathcal{P}^{\mathrm{ac}}$ will denote the set of measures in $\mathcal{P}$ absolutely continuous with respect to the Lebesgue measure and $\mathcal{P}^{\mathrm{fin}}$ will denote the set of measures in $\mathcal{P}$ with finite independent support.

LEMMA **4.2.** *Let $I_1, \dots, I_m$ be disjoint intervals of $\mathbb{T}$, $A = \bigcup\limits_{j=1}^{m} I_j$ and $\mu \in \mathcal{P} \cap \mathcal{M}(A)$. Then there exist sequences $\mu^n \in \mathcal{P}^{\mathrm{ac}}$ and $\nu^n \in \mathcal{P}^{\mathrm{fin}}$ $(n \in \mathbb{N})$ such that*

$$\mu^n(I_j) = \nu^n(I_j) = \mu(I_j) \ \text{ for } \ 1 \leqslant j \leqslant m \ \text{ and } \tag{22}$$

$$\mu^n \xrightarrow{\sigma} \mu, \ \nu^n \xrightarrow{\sigma} \nu, \ \mu^n_{I_j} \xrightarrow{\sigma} \mu_{I_j}, \ \nu^n_{I_j} \xrightarrow{\sigma} \mu_{I_j} \ \text{ as } n \to \infty \text{ for } 1 \leqslant j \leqslant m. \tag{23}$$

P r o o f. Let B be the set of atoms of μ, that is $B = \{z \in \mathbb{T} : \mu(\{z\}) > 0\}$, which is at most countable since μ is finite. Thus, for each interval J of T and $\varepsilon > 0$, there exists a disjoint family of intervals $J_1, \dots, J_d$ such that $\max\limits_{1 \leqslant j \leqslant d} |J_j| < \varepsilon$, $J_j \subset J$ for any $j = 1, \dots, d$ and $J \setminus \bigcup\limits_{j=1}^{d} J_j$ is finite and does not meet B. In this way it is easy to choose a sequence $\mathbb{I}^n = \{I_1^n, I_2^n, \dots, I_{k_n}^m\}$ of disjoint families of intervals of $\mathbb{T}$ with $\mathbb{I}^1 = \{I_1, \dots, I_m\}$, for which (L1) and (L2) are satisfied and for any $n \in \mathbb{N}$ and the set $\bigcup\limits_{j=1}^{k_n} I_j^n \setminus \bigcup\limits_{l=1}^{k_{n+1}} I_l^{n+1}$ is finite and does not meet B. The latter property implies that

$$\sum_{j=1}^{k_n} \mu(I_j^n) = 1 \ \text{ for each } \ n \in \mathbb{N}. \tag{24}$$

Now we can define μ^n and ν^n. Let μ^n be the absolutely continuous measure with the density

$$\rho^n(z) = \sum_{j=1}^{k_n} c_j^n \chi_{I_j^n},$$

where χ_I denotes the indicator function of a set I and $c_j^n \geqslant 0$ are chosen in such a way that $\mu^n(I_j^n) = \mu(I_j^n)$ for $1 \leqslant j \leqslant k_n$. Hence $\mu^n(I_j) = \mu(I_j)$ for $n \in \mathbb{N}$ and $1 \leqslant j \leqslant m$. From (24) it also follows that $\mu^n \in \mathcal{P}^{\rm ac}$.

To define ν^n, choose a set of independent points $z_{n,j} \in I_j^n$, $1 \leqslant j \leqslant k_n$ and consider

$$\nu^n = \sum_{j=1}^{k_n} \mu(I_j^n)\delta_{z_{n,j}},$$

where δ_z stands for the probability measure with the one-point support $\{z\}$. From (24), we see that $\nu^n \in \mathcal{P}$ and therefore $\nu^n \in \mathcal{P}^{\rm fin}$. Obviously $\nu^n(I_j^n) = \mu(I_j^n)$ for $1 \leqslant j \leqslant k_n$ and therefore $\nu^n(I_j) = \mu(I_j)$ for $j = 1, \dots, m$. Thus, (22) holds. Finally (23) follows from Lemma 4.1. □

Next lemma is the main building block in the inductive procedure of constructing the measure, asserted by Theorem 1.2.

Lemma **4.3.** *Let $\varepsilon > 0$, $k_0 \in \mathbb{N}$, $h_1, \dots, h_n \in \mathcal{C}(\mathbb{T})$, $I_1, \dots, I_m$ be disjoint intervals, $c_1, \dots, c_m \in \mathbb{C} \setminus \{0\}$ with $a = \max\{|c_1|, \dots, |c_m|\} \leqslant 1$ and $\mu \in \mathcal{P}^{\rm ac}$ be such that $\sum\limits_{j=1}^{m} \mu(I_j) = 1$. Then there exist $\nu \in \mathcal{P}^{\rm ac}$ and $k \in \mathbb{N}$, $k > k_0$ satisfying*

(B1) $\nu(I_j) = \mu(I_j)$ *for $1 \leqslant j \leqslant m$;*

(B2) $|[\mu - \nu, h_l]| < \varepsilon$ *for $1 \leqslant l \leqslant n$;*

(B3) $|\widehat{\nu_{I_j}}(k) - c_j\mu(I_j)| < \varepsilon$ *for $1 \leqslant j \leqslant m$;*

(B4) $\|\widehat{\mu} - \widehat{\nu}\|_\infty \leqslant 2a$.

Proof. Since μ_{I_j} are absolutely continuous, $\widehat{\mu_{I_j}}(k) \to 0$ as $|k| \to \infty$ for $1 \leqslant j \leqslant m$. Therefore there exists $k_1 \in \mathbb{N}$ such that $|\widehat{\mu_{I_j}}(k)| < \varepsilon/3$ if $|k| \geqslant k_1$ and $1 \leqslant j \leqslant m$. By Lemma 4.2, there exists $\gamma \in \mathcal{P}^{\rm fin}$ such that

$$\gamma(I_j) = \mu(I_j) \ \text{ for } \ 1 \leqslant j \leqslant m; \tag{25}$$

$$|[\mu_{I_j} - \gamma_{I_j}, h_l]| < \varepsilon/(2m) \ \text{ for } \ 1 \leqslant j \leqslant m \text{ and } 1 \leqslant l \leqslant n. \tag{26}$$

Since $\operatorname{supp}(\gamma) = \{u_1, \dots, u_N\}$ is an independent set, the Kronecker theorem [17] implies that the sequence $\{(u_1^k, \dots, u_N^k)\} : k \in \mathbb{N}\}$ is dense in $\mathbb{T}^N$. Consider the vector $\theta \in \mathbb{T}^N$ defined by

$$\theta_s = c_j/|c_j| \ \text{ if } \ u_s \in I_j.$$

Choosing $k > \max\{k_0, k_1\}$ in such a way that $(u_1^k, \dots, u_N^k)$ is close enough to θ, we can ensure that

$$|\widehat{\gamma_{I_j}}(k) - c_j\gamma(I_j)/|c_j|| = |\widehat{\gamma_{I_j}}(k) - c_j\mu(I_j)/|c_j|| < \varepsilon/3 \ \text{ for } \ 1 \leqslant j \leqslant m. \tag{27}$$

Applying Lemma 4.2 once again, we obtain that there exists $\eta \in \mathcal{P}^{\rm ac}$ such that

$$\eta(I_j) = \gamma(I_j) = \mu(I_j) \ \text{ for } \ 1 \leqslant j \leqslant m; \tag{28}$$

$$|[\eta_{I_j} - \gamma_{I_j}, g_l]| < \varepsilon/(2m) \ \text{ for } \ 1 \leqslant j \leqslant m \text{ and } 1 \leqslant l \leqslant n; \tag{29}$$

$$|\widehat{\eta_{I_j}}(k) - \widehat{\gamma_{I_j}}(k)| < \varepsilon/3 \ \text{ for } \ 1 \leqslant j \leqslant m. \tag{30}$$

The required measure is

$$\nu = \sum_{j=1}^{m} (1 - |c_j|)\mu_{I_j} + |c_j|\eta_{I_j}, \tag{31}$$

which, since $|c_j| \leqslant 1$, is non-negative and clearly absolutely continuous. From (28) we find that $\nu(I_j) = \mu(I_j)$ for $1 \leqslant j \leqslant m$. Thus, $\nu \in \mathcal{P}^{\mathrm{ac}}$ and (B1) holds. From (31) it follows that

$$|[\mu - \nu, g_l]| = \left|\sum_{j=1}^{m} |c_j|[\mu_{I_j} - \eta_{I_j}, g_l]\right| \leqslant a \sum_{j=1}^{m}(|[\eta_{I_j} - \gamma_{I_j}, g_l]| + |[\mu_{I_j} - \gamma_{I_j}, g_l]|).$$

Using (26) and (29), we obtain

$$|[\mu - \nu, g_l]| < am(\varepsilon/(2m) + \varepsilon/(2m)) \leqslant a\varepsilon \leqslant \varepsilon \ \text{ for } 1 \leqslant j \leqslant n.$$

Hence (B2) holds. Suppose now that $1 \leqslant j \leqslant m$. Then

$$\begin{aligned}|\widehat{\nu_{I_j}}(k) - c_j\mu(I_j)| &= |(1 - |c_j|)\widehat{\mu_{I_j}}(k) + |c_j|\widehat{\eta_{I_j}}(k) - c_j\mu(I_j)| \leqslant \\ &\leqslant |\widehat{\mu_{I_j}}(k)| + |c_j||\widehat{\eta_{I_j}}(k) - \widehat{\gamma_{I_j}}(k)| + |c_j||\widehat{\gamma_{I_j}}(k) - c_j\mu(I_j)/|c_j||.\end{aligned}$$

Thus by (30), (27) and the inequalities $|\widehat{\mu_{I_j}}(k)| < \varepsilon/3$, $|c_j| \leqslant 1$, we obtain $|\widehat{\nu_{I_j}}(k) - c_j\mu(I_j)| < \varepsilon$, that is, (B3) holds. Finally from (28) it follows that $\|\mu_{I_j}\| = \|\eta_{I_j}\| = \mu(I_j)$ and we have

$$\|\widehat{\mu} - \widehat{\nu}\|_\infty \leqslant \|\mu - \nu\| = \left\|\sum_{j=1}^{m} |c_j|(\mu_{I_j} - \eta_{I_j})\right\| \leqslant 2a \sum_{j=1}^{m} \mu(I_j) = 2a.$$

Thus, (B4) also holds. □

Lemma 4.4. *Let $\delta_n > 0$ and $f_n \in \mathcal{C}(\mathbb{T})$ $(n \in \mathbb{N})$ be such that $\|f_n\|_\infty \to 0$ as $n \to \infty$ and $\|f_n\|_\infty \leqslant 1$ for any $n \in \mathbb{N}$. Then there exists $\mu \in \mathcal{P} \cap \mathcal{M}_0$ and a strictly increasing sequence k_n of non-negative integers such that*

$$|[\mu, g_n\overline{g_m}]| \leqslant \delta_n \quad \textit{whenever} \quad n > m, \tag{32}$$

where $g_n(z) = z^{k_n} - f_n(z)$.

Proof. First of all, we take a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive numbers such that

$$\sum_{k=n}^{\infty} \varepsilon_k < \delta_n/6 \ \text{ for each } \ n \in \mathbb{N}. \tag{33}$$

For any $n, k \in \mathbb{N}$, $1 \leqslant k \leqslant n$, let I_k^n be the interval of $\mathbb{T}$ between $e^{2\pi i(k-1)/n}$ and $e^{2\pi ik/n}$ (going counterclockwise). Obviously, for any fixed n, the intervals $I_1^n, \dots, I_n^n$ are disjoint and $\mathbb{T} \setminus \bigcup_{j=1}^{n} I_j^n$ is finite. Therefore $\nu = \sum_{k=1}^{n} \nu_{I_k^n}$ for any $n \in \mathbb{N}$ and any continuous measure $\nu \in \mathcal{M}$.

Set $a_n = \|f_n\|_\infty$. For each $n \in \mathbb{N}$ we shall construct inductively non-negative integers k_n, j_n, m_n, complex numbers $c_j^{n,d}, b_j^n$ $(1 \leqslant d \leqslant n,\ 1 \leqslant j \leqslant m_n)$ and a measure $\mu^n \in \mathcal{P}^{\mathrm{ac}}$, satisfying the following conditions:

(P1) $0 \leqslant k_{n-1} < k_n$, $1 \leqslant j_{n-1} < j_n$ and $1 \leqslant m_{n-1} < m_n$ if $n \geqslant 2$;
(P2) m_{n-1} is a divisor of m_n for $n \geqslant 2$;
(P3) $|c_j^{n,d}| \leqslant 2$, $|b_j^n| \leqslant a_{n+1}$, $|g_d(z) - c_j^{n,d}| \leqslant \varepsilon_{n+1}$ and $|f_{n+1}(z) - b_j^n| \leqslant \varepsilon_{n+1}$ for $1 \leqslant d \leqslant n$, $1 \leqslant j \leqslant m_n$ and $z \in I_j^{m_n}$, where $g_d(z) = z^{k_d} - f_d(z)$;
(P4) $\mu^n(I_j^{m_{n-1}}) = \mu^{n-1}(I_j^{m_{n-1}})$ for $1 \leqslant j \leqslant m_{n-1}$ and $n \geqslant 2$;

(P5) $|\widehat{\mu_j^n}(k_n) - b_j^{n-1}\mu^n(I_j^{m_{n-1}})| \leqslant \varepsilon_n/m_{n-1}$ for $1 \leqslant j \leqslant m_{n-1}$ and $n \geqslant 2$, where μ_j^n is the restriction of μ^n to $I_j^{m_{n-1}}$;

(P6) $|\widehat{\mu^n}(l)| \leqslant \varepsilon_n$ for $|l| \geqslant j_n$;

(P7) $|\widehat{\mu^n}(l) - \widehat{\mu^{n-1}}(l)| \leqslant \varepsilon_n$ for $|l| < j_{n-1}$ if $n \geqslant 2$;

(P8) $\|\widehat{\mu^n} - \widehat{\mu^{n-1}}\|_\infty \leqslant 2a_{n+1}$ for $n \geqslant 2$;

(P9) $|[\mu^n - \mu^{n-1}, g_m\overline{g_l}]| < \varepsilon_n$ for $1 \leqslant l < m \leqslant n-1$ and $n \geqslant 2$.

Set $k_1 = 0$ and $g_1(z) = z^{k_1} - f_1(z) = 1 - f_1(z)$. Take an arbitrary measure $\mu^1 \in \mathcal{P}^{\rm ac}$. Since the functions f_2 and g_1 are uniformly continuous, there exist $m_1 \in \mathbb{N}$ and complex numbers $c_j^{1,1}$, b_j^1 $(1 \leqslant j \leqslant m_1)$ such that $|c_j^{1,1}| \leqslant \|g_1\|_\infty \leqslant 1 + a_1 \leqslant 2$, $|b_j^n| \leqslant \|f_2\|_\infty = a_2$ and $|g_1(z) - c_j^{1,1}| \leqslant \varepsilon_2$, $|f_2(z) - b_j^1| \leqslant \varepsilon_2$ if $1 \leqslant j \leqslant m_1$ and $z \in I_j^{m_1}$. Since μ^1 is absolutely continuous, there exists $j_1 \in \mathbb{N}$ such that $|\widehat{\mu^1}(l)| \leqslant \varepsilon_1$ for $|l| \geqslant j_1$. Thus, k_1, j_1, m_1, $c_j^{1,1}, b_j^1$ and μ^1 satisfy (P3) and (P6): the only conditions required for $n = 1$. The first step (basis) of induction is done.

Assume now that $n \geqslant 2$ and k_l, j_l, m_l, $c_j^{l,d}$, b_j^l and μ^l for $1 \leqslant l \leqslant n-1$, $1 \leqslant d \leqslant l$ and $1 \leqslant j \leqslant m_l$, satisfying (P1–P9) are already constructed. We have to construct k_n, j_n, m_n, $c_j^{n,d}$, b_j^n and μ^n.

By Lemma 4.3 applied for $\mu = \mu^{n-1}$, $I_j = I_j^{m_{n-1}}$, $c_j = b_j^{n-1}$, $k_0 = k_{n-1}$, $m = m_{n-1}$, $\varepsilon = \varepsilon_n/m_{n-1}$ and the finite set of functions $\{h_j\}$ being $\{g_m\overline{g_l} : 1 \leqslant l < m \leqslant n-1\} \cup \{z^l : |l| < j_{n-1}\}$, there exists a measure $\mu^n \in \mathcal{P}^{\rm ac}$ and $k_n \in \mathbb{N}$ such that $k_n > k_{n-1}$ and (P4), (P5), (P7), (P8) and (P9) are satisfied.

Since f_{n+1} and $g_d(z) = z^{k_d} - f_d(z)$ $(1 \leqslant d \leqslant n)$ are uniformly continuous and $\|g_d\|_\infty \leqslant 1 + a_d \leqslant 2$, $\|f_{n+1}\|_\infty \leqslant a_{n+1}$, there exist $m_n \in \mathbb{N}$ and complex numbers $c_j^{n,d}$, b_j^n $(1 \leqslant d \leqslant n,\ 1 \leqslant j \leqslant m_n)$ such that $m_n > m_{n-1}$, m_{n-1} is a divisor of m_n and (P3) is satisfied. Since μ^n is absolutely continuous, there exists $j_n \in \mathbb{N}$ such that $j_n > j_{n-1}$ and (P6) is satisfied. Obviously conditions (P1) and (P2) are also satisfied.

Thus, the induction step is described and the construction of k_n, j_n, m_n, $c_j^{n,d}, b_j^n$ and μ^n is complete.

First, we shall prove weak convergence of μ^n to a measure $\mu \in \mathcal{P}$. Let $f \in \mathcal{C}(\mathbb{T})$ and $\varepsilon > 0$. Since $m_n \to \infty$ as $n \to \infty$ and f is uniformly continuous, there exist $a \in \mathbb{N}$ and complex numbers $z_1, \dots, z_{m_a}$ such that $|f(z) - z_j| \leqslant \varepsilon$ if $z \in I_j^a$ and $1 \leqslant j \leqslant m_a$. From (P2) and (P4) it follows that $\mu^n(I_j^a) = \mu^m(I_j^a)$ if $m \geqslant a$, $n \geqslant a$ and $1 \leqslant j \leqslant m_a$. The same argument as in the proof of Lemma 4.1 shows that $[\mu^n - \mu^m, f] \leqslant 2\varepsilon$ for any $m, n \geqslant a$. Hence μ^n is a Cauchy sequence with respect to σ. Since, according to the Prokhorov theorem [8], $\mathcal{P}$ is compact in $(\mathcal{M}, \sigma)$, there exists $\mu \in \mathcal{P}$ such that $\mu^n \xrightarrow{\sigma} \mu$ as $n \to \infty$.

Next, we shall show that μ together with the sequence k_n satisfy the statement of the Lemma. First, we prove that $\mu \in \mathcal{M}_0$. Let $n \in \mathbb{N}$ and $l \in \mathbb{Z}$ be such that $j_n \leqslant |l| < j_{n+1}$. By (P6) $|\widehat{\mu^n}(l)| \leqslant \varepsilon_n$. According to (P7) we have $|\widehat{\mu^{k+1}}(l) - \widehat{\mu^k}(l)| \leqslant \varepsilon_{k+1}$ for $k \geqslant n+1$. Finally, (P8) implies that $|\widehat{\mu^{n+1}}(l) - \widehat{\mu^n}(l)| \leqslant 2a_{n+2}$. Thus,

$$|\widehat{\mu}(l)| \leqslant |\widehat{\mu^n}(l)| + |\widehat{\mu^{n+1}}(l) - \widehat{\mu^n}(l)| + \sum_{k=n+1}^{\infty} |\widehat{\mu^{k+1}}(l) - \widehat{\mu^k}(l)| \leqslant 2a_{n+2} + \sum_{j=n}^{\infty} \varepsilon_j \to 0$$

as $n \to \infty$. Therefore $\widehat{\mu}(l) \to 0$ as $|l| \to \infty$, that is $\mu \in \mathcal{M}_0$.

It remains to estimate $[\mu, g_n\overline{g_d}]$. Let $d, n \in \mathbb{N}$ and $d < n$. Denote $h_n(z) = z^{k_n}$. Since $g_l = h_l - f_l$, we can write

$$g_n\overline{g_d} = (h_n - b_j^{n-1})\overline{c_j^{n-1,d}} + (b_j^{n-1} - f_n)\overline{c_j^{n-1,d}} + g_n(\overline{g_d} - \overline{c_j^{n-1,d}}).$$

Taking into account that $\mu^n = \sum\limits_{j=1}^{m_{n-1}} \mu_j^n$, where μ_j^n is the restriction of μ^n to $I_j^{m_{n-1}}$, we obtain

$$[\mu^n, g_n\overline{g_d}] = \sum_{j=1}^{m_{n-1}} [\mu_j^n, g_n\overline{g_d}] = \sum_{j=1}^{m_{n-1}} (A_j^n + B_j^n + C_j^n), \quad \text{where}$$
$$A_j^n = [\mu_j^n, (h_n - b_j^{n-1})\overline{c_j^{n,d}}], \;\; B_j^n = [\mu_j^n, (b_j^{n-1} - f_n)\overline{c_j^{n,d}}] \text{ and } C_j^n = [\mu_j^n, g_n(\overline{g_d} - \overline{c_j^{n,d}})].$$

Using that $\|g_n\|_\infty \leqslant 1 + a_n \leqslant 2$ and (P3), we find that $|c_j^{n,d}| \leqslant 2$ and $|b_j^{n-1} - f_n(z)| \leqslant \varepsilon_n$ and $|\overline{g_d}(z) - \overline{c_j^{n,d}}| \leqslant \varepsilon_n$ for $z \in I_j^{m_{n-1}}$. Since μ_j^n is supported on $I_j^{m_{n-1}}$, we have

$$|B_j^n| \leqslant 2\varepsilon_n \mu^n(I_j^{m_{n-1}}) \;\; \text{and} \;\; |C_j^n| \leqslant 2\varepsilon_n \mu^n(I_j^{m_{n-1}}).$$

On the other hand

$$A_j^n = \overline{c_j^{n,d}}([\mu_j^n, h_n] - b_j^{n-1}\mu^n(I_j^{m_{n-1}})) = \overline{c_j^{n,d}}(\widehat{\mu_j^n}(k_n) - b_j^{n-1}\mu^n(I_j^{m_{n-1}})).$$

Using (P5) and the fact that $|c_j^{n,d}| \leqslant 2$, we obtain

$$|A_j^n| \leqslant 2\varepsilon_n / m_{n-1}.$$

Upon putting the estimates on A_j^n, B_j^n and C_j^n together, we have

$$|[\mu^n, g_n\overline{g_d}]| \leqslant \sum_{j=1}^{m_{n-1}} (|A_j^n| + |B_j^n| + |C_j^n|) \leqslant 4\varepsilon_n \left(\sum_{j=1}^{m_{n-1}} \mu^n(I_j^{m_{n-1}}) \right) + 2m_{n-1}\varepsilon_n / m_{n-1} = 6\varepsilon_n.$$

From (P.9) for $m \geqslant n > d$ it follows that

$$|[\mu^{m+1} - \mu^m, g_n\overline{g_d}]| \leqslant \varepsilon_{m+1}.$$

Therefore, from (33) we see that

$$|[\mu, g_n\overline{g_d}]| \leqslant |[\mu^n, g_n\overline{g_d}]| + \sum_{m=n}^{\infty} |[\mu^{m+1} - \mu^m, g_n\overline{g_d}]| \leqslant 6\varepsilon_n + \sum_{m=n}^{\infty} \varepsilon_{m+1} < 6 \sum_{m=n}^{\infty} \varepsilon_m < \delta_n.$$

Thus, μ satisfies all required conditions. □

4.3 Proof of Theorem 1.2

We choose a set $S = \{h_n : n \in \mathbb{N}\}$ dense in the unit sphere of the Banach space $\mathcal{C}(\mathbb{T})$ and a one-to-one map $\varphi = (\varphi_1, \varphi_2)$ from $\mathbb{N}$ onto $\mathbb{N}^2$. We consider

$$f_n = 2^{-\varphi_1(n)}(\varphi_2(n))^{-1/2} h_{\varphi_1(n)},$$

which are in $\mathcal{C}(\mathbb{T})$, $\|f_n\|_\infty < 1$ for any $n \in \mathbb{N}$ and $\|f_n\|_\infty \to 0$ as $n \to \infty$. By Lemma 4.4 there exists $\mu \in \mathcal{P} \cap \mathcal{M}_0$ and a strictly increasing sequence k_n of positive integers, such that

$$|[\mu, g_n\overline{g_m}]| \leqslant 2^{-n} \;\; \text{whenever } n > m,$$

where $g_n(z) = z^{k_n} - f_n(z)$. Since S is norm-dense in the unit sphere of $\mathcal{C}(\mathbb{T})$, we find that $\Omega = \{\lambda x : \lambda \in \mathbb{C}, \; x \in S\}$ is norm-dense in $\mathcal{C}(\mathbb{T})$, which is in turn norm dense in $L_2(\mu)$. It follows that Ω is weakly dense in $L_2(\mu)$,

Since $|[\mu, g_n\overline{g_m}]| = \langle g_n, g_m\rangle$, where $\langle\cdot,\cdot\rangle$ stands for the inner product in $L_2(\mu)$, we have

$$\sum_{\substack{m,n\in\mathbb{N}\\ n>m}} |\langle g_n, g_m\rangle|^2 \leqslant \sum_{n=2}^{\infty}(n-1)4^{-n} < \infty.$$

According to Lemma 2.6, $\{g_n\}$ is a 2-sequence in $L_2(\mu)$.

We shall show that the constant function $u(z) \equiv 1$ is a weakly supercyclic vector for the operator $Mf(z) = zf(z)$ acting on $L_2(\mu)$. For $n \in \mathbb{N}$, let $A_n = \{m \in \mathbb{N} : \varphi_1(m) = n\}$. Since φ is one-to-one from $\mathbb{N}$ onto $\mathbb{N}^2$, it follows that φ_2 is one-to-one from A_n onto $\mathbb{N}$. Let $m \in A_n$. We have $f_m = 2^{-n}(\varphi_2(m))^{-1/2}h_n$ and $g_m = T^{k_m}u - f_m$. Hence

$$g_m = \beta(m)T^{\gamma(m)} - \alpha(m)h_n \ \text{ for each } \ m \in A_n,$$

where $\beta(m) = 1$, $\gamma(m) = k_m$ and $\alpha(m) = 2^{-n}(\varphi_2(m))^{-1/2}$. Since φ_2 is one-to-one from A_n onto $\mathbb{N}$, we obtain

$$\sum_{m\in A_n} |\alpha(m)|^2 = \sum_{m\in A_n} \bigl(2^n(\varphi_2(m))^{1/2}\bigr)^{-2} = 2^{-2n}\sum_{j=1}^{\infty} j^{-1} = \infty.$$

Upon applying Proposition 2.4, we see that u is a weakly supercyclic vector for M. Clearly the requirement $\mu \in \mathcal{P}\cap\mathcal{M}_0$ also holds. The proof of Theorem 1.2 is complete.

5 Proof of Theorem 1.6

We start with reformulating the Salas criteria of hypercyclicity and supercyclicity of bilateral weighted shift in a more convenient form. This form is reminiscent of the one of Feldman [14], which he obtained under the additional assumption of invertibility.

Proposition **5.1.** *Let T be a bilateral weighted shift acting on $\ell_p(\mathbb{Z})$ with $1 \leqslant p < \infty$ or $c_0(\mathbb{Z})$. Then T is hypercyclic if and only if for any $k \in \mathbb{N}_0$,*

$$\varliminf_{n\to\infty} \max\{\beta(k-n+1,k), (\beta(k+1,k+n))^{-1}\} = 0 \tag{34}$$

and T is supercyclic if and only if for any $k \in \mathbb{N}_0$,

$$\varliminf_{n\to+\infty} \beta(k-n+1,k)\beta(k+1,k+n)^{-1} = 0, \tag{35}$$

where $\beta(a,b)$ are the numbers defined in (1).

P r o o f. Obviously, if (2) is satisfied for any $k \in \mathbb{N}_0$ then (34) holds true for any $k \in \mathbb{N}_0$ and if (3) is satisfied for any $k \in \mathbb{N}_0$ then (35) holds true for any $k \in \mathbb{N}_0$. It remains to prove the opposite. For any $m \in \mathbb{N}$ denote

$$d_m = \bigl(\max\{1, \|w\|_\infty\}\bigr)^{-2m} \min_{-m\leqslant a\leqslant b\leqslant m} \beta(a,b).$$

Suppose that (34) holds for any $k \in \mathbb{N}_0$ and (2) fails for $k = m-1 \in \mathbb{N}_0$. Then there exist sequences $\{j_n\}_{n\in\mathbb{N}_0}$, $\{k_n\}_{n\in\mathbb{N}_0}$ and $c > 0$ such that $|j_n| < m$, $|k_n| < m$ and $\max\{\beta(j_n - n, j_n), (\beta(k_n, k_n+n))^{-1}\} \geqslant c$ for each $n \in \mathbb{N}_0$. Since

$$\beta(m-n+1,m) = \frac{\beta(j_n-n,j_n)\beta(j_n+1,m)}{\beta(j_n-n,m-n)} \geqslant d_m\beta(j_n-n,j_n) \ \text{ and} \tag{36}$$

$$\beta(m+1,m+n) \leqslant \frac{\beta(k_n,k_n+n)\beta(k_n+n+1,m+n)}{\beta(k_n,m)} \leqslant \frac{\beta(k_n,k_n+n)}{d_m}, \tag{37}$$

we obtain that

$$\max\{\beta(m-n+1,m),(\beta(m+1,m+n))^{-1}\} \geqslant d_m \max\{\beta(j_n-n,j_n),(\beta(k_n,k_n+n))^{-1}\} \geqslant cd_m$$

for each $n \in \mathbb{N}_0$. Thus, (34) fails for $k=m$. A contradiction.

Finally suppose that (35) holds for any $k \in \mathbb{N}_0$ and (3) fails for $k=m-1 \in \mathbb{N}_0$. Then there exist sequences $\{j_n\}_{n\in\mathbb{N}_0}$, $\{k_n\}_{n\in\mathbb{N}_0}$ and $c>0$ such that $|j_n|<m$, $|k_n|<m$ and $\frac{\beta(j_n-n,j_n)}{\beta(k_n,k_n+n)} \geqslant c$ for each $n \in \mathbb{N}_0$. Applying (36) and (37), we obtain

$$\beta(m-n,m)\beta(m,m+n)^{-1} \geqslant d_m^2 \beta(j_n-n,j_n)\beta(k_n,k_n+n)^{-1} \geqslant d_m^2 c$$

for any $n \in \mathbb{N}_0$. Thus, (35) fails for $k=m$. A contradiction. □

The proof is based on the following two propositions on weak closeness of sequences in ℓ_p with rapidly increasing norms.

PROPOSITION **5.2.** *Let $\mathcal{H}$ be a real or complex Hilbert space and $\{x_n\}_{n\in\mathbb{N}_0}$ be a sequence of elements of $\mathcal{H}$, such that*

$$\sum_{n=0}^{\infty} \|x_n\|^{-a} < \infty \tag{38}$$

for $a=2$. Then the set $S=\{x_n : n \in \mathbb{N}_0\}$ is weakly closed in $\mathcal{H}$.

PROPOSITION **5.3.** *Let $1<p<\infty$ and $\{x_n\}_{n\in\mathbb{N}_0}$ be a sequence of elements of the real or complex Banach space $\ell_p(\Lambda)$, such that (38) is satisfied with $0<a<\min\{2,q\}$, where $\frac{1}{p}+\frac{1}{q}=1$. Then the set $S=\{x_n : n \in \mathbb{N}_0\}$ is weakly closed in $\ell_p(\Lambda)$.*

We shall now prove Theorems 1.6 with the help of these results, postponing the proofs of the propositions to the end of the section. For sake of completeness we formulate an analog of Propositions 5.2 and 5.3 for general Banach spaces, which we also prove in the end of the section.

PROPOSITION **5.4.** *Let $\mathcal{B}$ be a real or complex Banach space and $\{x_n\}_{n\in\mathbb{N}_0}$ be a sequence of elements of $\mathcal{B}$, such that (38) is satisfied with $a=1$. Then the set $S=\{x_n : n \in \mathbb{N}_0\}$ is weakly closed in $\mathcal{B}$.*

Note that weak closeness of a countable subset $\{x_n : n \in \mathbb{N}_0\}$ of a Banach space under the condition that $\|x_n\|$ grow exponentially was proved in [11].

5.1 Proof of Theorem 1.6: the supercyclicity case

In this section $\mathbb{K}$ stands for either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers.

LEMMA **5.5.** *Let $\{x_n\}_{n\in\mathbb{N}_0}$ be a sequence in a Banach space $\mathcal{B}$ over the field $\mathbb{K}$, $y \in \mathcal{B}$, $z \in \mathcal{B}^*$ be such that $\langle y,z\rangle = 1$ and $\Omega=\{\lambda x_n : \lambda \in \mathbb{K},\ n \in \mathbb{N}_0\}$. If y belongs to the weak closure of Ω, then it belongs to the weak closure of*

$$N=\Bigl\{\frac{x_n}{\langle x_n,z\rangle} : n \in \mathbb{N}_0,\quad \langle x_n,z\rangle \neq 0\Bigr\}.$$

P r o o f. Let $\mathcal{B}_0=\{u \in \mathcal{B} : \langle u,z\rangle = 0\}$ and consider $M=\mathcal{B}\setminus\mathcal{B}_0$, $\Omega_0=\Omega\cap\mathcal{B}_0$ and $\Omega_1=\Omega\setminus\mathcal{B}_0$. Clearly, $\Omega=\Omega_0\cup\Omega_1$ and y is not in the weak closure of Ω_0, since Ω_0 is contained in the weakly closed set $\mathcal{B}_0$ and $y \notin \mathcal{B}_0$. Hence, y is in the weak closure of Ω_1. Since the map

$$F: M \to \mathcal{B},\quad F(u)=\frac{u}{\langle u,z\rangle}$$

is weak-to-weak continuous and y is in the weak closure of Ω_1, we obtain that $F(y) = y$ is in the weak closure of $F(\Omega_1) = N$, as required. □

We start with a general condition for an operator to be not weakly supercyclic.

THEOREM **5.6.** *Let T be a bounded linear operator acting on an infinite dimensional Banach space $\mathcal{B}$ and $f \in \mathcal{B}$ be such that $T^n f \neq 0$ for each $n \in \mathbb{N}_0$. Assume that there exists $y \in \mathcal{B}^*$, $y \neq 0$ and $a > 0$ for which*

$$\sum_{n=0}^{\infty} \left(\frac{|\langle T^n f, y\rangle|}{\|T^n f\|} \right)^a < \infty. \tag{39}$$

Suppose also that either $a = 1$ or $\mathcal{B}$ is a Hilbert space and $a = 2$ or $\mathcal{B}$ is isomorphic to ℓ_p with $1 < p < \infty$ and $a < \min\{2, q\}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then f is not a weakly supercyclic vector for T.

Proof. Since $\mathcal{B}$ is infinite dimensional, we can pick $x \in \mathcal{B} \setminus O_{\rm pr}(T, f)$ such that $\langle x, y\rangle = 1$. Suppose that f is a weakly supercyclic vector for T. Then x is in the weak closure of $O_{\rm pr}(T, f)$. By Lemma 5.5 x is in the weak closure of the set

$$N = \{u_n : n \in A\}, \quad \text{where} \quad A = \{n \in \mathbb{N}_0 : \langle T^n f, y\rangle \neq 0\} \quad \text{and} \quad u_n = \frac{T^n f}{\langle T^n f, y\rangle}.$$

From (39) it follows that $\sum\limits_{n \in A} \|u_n\|^{-a} < \infty$. Applying Propositions 5.2, 5.3 and 5.4, we see that N is weakly closed in $\mathcal{B}$. Hence the only way for x to be in the weak closure of N is to coincide with one of the u_n's. In this case $x \in O_{\rm pr}(T, f)$. A contradiction. □

Now we are ready to prove the supercyclicity part of Theorem 1.6. We have to demonstrate that any weakly supercyclic bilateral weighted shift acting on $\ell_p(\mathbb{Z})$ with $1 \leqslant p \leqslant 2$ is supercyclic. Since according to Theorem S, supercyclicity of a bilateral weighted shift does not depend on p, by comparison principle it suffices to consider the case $p = 2$. Suppose that T is a bilateral weighted shift acting on $\ell_2(\mathbb{Z})$, which is weakly supercyclic and non-supercyclic, w is its weight sequence and $\beta(a, b)$ are the numbers defined in (1). By Proposition 5.1 there are $c > 0$ and $m \in \mathbb{N}_0$ such that

$$\beta(m - n + 1, m) \geqslant c\,\beta(m + 1, m + n) \quad \text{for each } n \in \mathbb{N}_0. \tag{40}$$

Since the set of weakly supercyclic vectors of a weakly supercyclic operator is weakly dense, there exists a weakly supercyclic vector x for T in $\ell_2(\mathbb{Z})$ such that $\langle x, e_m\rangle \neq 0$. Using (40), we have

$$\frac{|\langle T^n x, e_m\rangle|}{\|T^n x\|_2} \leqslant \frac{|\langle x, T^{*n} e_m\rangle|}{|\langle x, e_m\rangle| \|T^n e_m\|_2} = \frac{|\langle x, e_{n+m}\rangle| \beta(m + 1, m + n)}{|\langle x, e_m\rangle| \beta(m - n + 1, m)} \leqslant \frac{|\langle x, e_{n+m}\rangle|}{c|\langle x, e_m\rangle|}.$$

Since $x \in \ell_2(\mathbb{Z})$, we see that $\sum\limits_{n=0}^{\infty} \left(\frac{|\langle T^n x, e_m\rangle|}{\|T^n x\|_2} \right)^2 < \infty$. By Theorem 5.6 x can not be a weakly supercyclic vector for T. A contradiction. The proof is complete.

5.2 Proof of Theorem 1.6: the hypercyclicity case

We start with the following lemma dealing with positive infinite matrices.

LEMMA **5.7.** *Let Λ be an infinite countable set and $\{a_{\alpha,\beta}\}_{\alpha,\beta \in \Lambda}$ be an infinite matrix with non-negative entries such that* $\max\{a_{\alpha,\beta}, a_{\beta,\alpha}\} \geqslant 1$ *for each $\alpha, \beta \in \Lambda$. Then*

$$\sum_{\alpha \in \Lambda} S_\alpha^{-r} < \infty \quad \textit{for each } r > 1, \textit{ where } \; S_\alpha = \sum_{\beta \in \Lambda} a_{\alpha,\beta} \in [0, \infty].$$

P r o o f. Let $M > 0$ and $\alpha_1, \dots, \alpha_n$ be pairwise different elements of Λ such that $S_{\alpha_j} \leqslant M$ for $1 \leqslant j \leqslant n$. Then

$$Mn \geqslant \sum_{j=1}^{n} S_{\alpha_j} = \sum_{\substack{1\leqslant j\leqslant n\\ \beta\in\Lambda}} a_{\alpha_j,\beta} \geqslant \sum_{1\leqslant j,k\leqslant n} a_{\alpha_j,\alpha_k} = \frac{1}{2}\sum_{1\leqslant j,k\leqslant n} (a_{\alpha_j,\alpha_k} + a_{\alpha_k,\alpha_j}).$$

Since $a_{\alpha,\beta} + a_{\beta,\alpha} \geqslant \max\{a_{\alpha,\beta}, a_{\beta,\alpha}\} \geqslant 1$ for each $\alpha, \beta \in \Lambda$, we obtain $Mn \geqslant n^2/2$. Hence $n \leqslant 2M$. Therefore for any $M > 0$ there exists at most $[2M]$ elements α of Λ for which $S_\alpha \leqslant M$, where $[t]$ stands for the integer part of $t \in \mathbb{R}$. It follows that there exists a bijection $\varphi : \mathbb{N} \to \Lambda$ such that the sequence $S_{\varphi(n)}$ is monotonically non-decreasing. Using the above estimate with $M = S_{\varphi(n)}$, we obtain that $S_{\varphi(n)} \geqslant n/2$ for each $n \in \mathbb{N}$. Hence

$$\sum_{\alpha\in\Lambda} S_\alpha^{-r} = \sum_{n=1}^{\infty} S_{\varphi(n)}^{-r} \leqslant \sum_{n=1}^{\infty} (n/2)^{-r} < \infty \ \text{ if } r > 1. \quad \Box$$

Now we are ready to prove the hypercyclicity part of Theorem 1.6. We have to demonstrate that any weakly hypercyclic bilateral weighted shift actin on $\ell_p(\mathbb{Z})$ with $1 \leqslant p < 2$ is hypercyclic. Since according to Theorem S, hypercyclicity of a bilateral weighted shift does not depend on p, by comparison principle it suffices to consider the case $1 < p < 2$. Suppose that T is a non-hypercyclic weakly hypercyclic bilateral weighted shift acting on $\ell_p(\mathbb{Z})$ with $1 < p < 2$, w is its weight sequence and $\beta(a, b)$ are the numbers defined in (1). By Proposition 5.1 there are $c \in (0, 1]$ and $m \in \mathbb{N}_0$ such that

$$\max\{\beta(m - n + 1, m), \beta(m + 1, m + n)^{-1}\} \geqslant c \ \text{ for each } n \in \mathbb{N}_0. \tag{41}$$

Let x be a weakly hypercyclic vector for T and

$$A = \{k \in \mathbb{N} : |\langle T^k x, e_m\rangle| > 1\}.$$

The set $\{T^k x : k \in A\}$ can not be weakly closed. Indeed, otherwise $O(T, x)$ can not be weakly dense in the non-empty weakly open set $\{u \in \ell_p(\mathbb{Z}) : |\langle u, e_m\rangle| > 1\}$. By Proposition 5.3,

$$\sum_{k\in A} \|T^k x\|_p^{-a} = \infty \ \text{ for each } a < 2. \tag{42}$$

By definition of the set A, we have $|\langle x, e_{k+m}\rangle|\beta(m + 1, k + m) > 1$ for any $k \in A$. Hence

$$|\langle x, e_{k+m}\rangle| > \beta(m + 1, m + k)^{-1} \ \text{ for each } k \in A. \tag{43}$$

Let now $j \in A$. Obviously

$$\|T^j x\|_p^p = \sum_{n\in\mathbb{Z}} \beta(n - j + 1, n)^p |\langle x, e_n\rangle|^p \geqslant \sum_{k\in A} \beta(m + k - j + 1, m + k)^p |\langle x, e_{m+k}\rangle|^p.$$

Using (43), we obtain

$$\|T^j x\|_p^p \geqslant \sum_{k\in A} \frac{\beta(m + k - j + 1, m + k)^p}{\beta(m + 1, m + k)^p} = c^p \sum_{k\in A} a_{j,k}, \tag{44}$$

$$\text{where } \ a_{j,k} = \begin{cases} c^{-p} & \text{if } k = j; \\ c^{-p}\beta(m + k - j + 1, m)^p & \text{if } k < j; \\ c^{-p}\beta(m + 1, m + k - j)^{-p} & \text{if } k > j. \end{cases}$$

From (41) it follows that $\max\{a_{j,k}, a_{k,j}\} \geqslant 1$ for each $j, k \in A$. Lemma 5.7 together with (44) implies that

$$\sum_{j\in A} \|T^j x\|_p^{-rp} < \infty \text{ for each } r > 1.$$

Since $p < 2$, we can choose $r > 1$ such that $rp < 2$. Hence the last display contradicts (42). The proof is complete.

5.3 Proof of Propositions 5.3 and 5.5

We need the following interesting theorems by Ball [3, 4].

Theorem **B1.** *Let $\mathcal{H}$ be a complex Hilbert space, $\{x_n\}_{n\in\mathbb{N}_0}$ be a sequence of elements of $\mathcal{H}$ such that $\|x_n\| = 1$ for any $n \in \mathbb{N}_0$ and $\{s_n\}_{n\in\mathbb{N}_0}$ be a sequence of positive numbers such that $\sum\limits_{n=0}^{\infty} s_n^2 = 1$. Then there exists $y \in \mathcal{H}$ such that $|\langle x_n, y\rangle| \geqslant s_n$ for each $n \in \mathbb{N}_0$.*

Theorem **B2.** *Let $\mathcal{B}$ be a real Banach space, $\{x_n\}_{n\in\mathbb{N}_0}$ be a sequence of elements of $\mathcal{B}$ such that $\|x_n\| = 1$ for any $n \in \mathbb{N}_0$ and $\{s_n\}_{n\in\mathbb{N}_0}$ be a sequence of positive numbers such that $\sum\limits_{n=0}^{\infty} s_n < 1$. Then there exists $y \in \mathcal{B}^*$ such that $|\langle x_n, y\rangle| \geqslant s_n$ for each $n \in \mathbb{N}_0$.*

The real and complex versions of Propositions 5.2 and 5.4 are equivalent to each other. Indeed the real case reduces to the complex one by replacing the space with its complexification and the complex case reduces to the real one just by considering the complex space as real. Thus, it suffices to prove Proposition 5.2 in the complex case and Proposition 5.3 in the real case. Let either $\mathcal{B}$ be a real Banach space or $\mathcal{B} = \mathcal{H}$ be a complex Hilbert space. Let $y \in \mathcal{B} \setminus S$ and $y_n = x_n - y$, $s_n = \|y_n\|^{-1}$ for $n \in \mathbb{N}_0$. In the Banach space case from (38) with $a = 1$ it follows that $\sum\limits_{n=0}^{\infty} s_n = C/2 < \infty$. In the Hilbert space case from (38) with $a = 2$ it follows that $\sum\limits_{n=0}^{\infty} s_n^2 = C^2 < \infty$. Applying Theorem B2 in the Banach space case and Theorem B1 in the Hilbert space case, we obtain that there exists $u \in \mathcal{B}^*$ with $\|u\| = 1$ such that $|\langle y_n/\|y_n\|, u\rangle| \geqslant s_n/C$ for $n \in \mathbb{N}_0$. Hence, $|\langle y_n, u\rangle| \geqslant C^{-1}$ for each $n \in \mathbb{N}_0$. It means that zero is not in the weak closure of $\{y_n : n \in \mathbb{N}_0\}$, or equivalently, y is not in the weak closure of S. Since y is an arbitrary point in $\mathcal{B} \setminus S$, we see that S is weakly closed.

5.4 Proof of Proposition 5.3

The ideal way to prove Proposition 5.3 would be to use an analog of Ball's theorem for ℓ_p-spaces. Unfortunately, it remains undiscovered. We use probabilistic approach to prove Proposition 5.3.

Recall few definitions. Let $\mathcal{B}$ be a real Banach space and $\mathcal{F}$ be the set of linearly independent finite subsets $Y = \{y_1, \dots, y_n\}$ of $\mathcal{B}^*$. Let $\mathcal{R}_Y$ denote the family of sets of the form

$$\{x \in \mathcal{B} : (\langle x, y_1\rangle, \dots, \langle x, y_n\rangle) \in B\}, \text{ where } B \text{ is a Borel subset of } \mathbb{R}^n.$$

Obviously, $\mathcal{R}_Y$ is a sub-sigma-algebra of the Borel sigma-algebra of $\mathcal{B}$. A cylindric set is any element of

$$\mathcal{R}(\mathcal{B}) = \bigcup_{Y\in\mathcal{F}} \mathcal{R}_Y.$$

Note that $\mathcal{R}(\mathcal{B})$ is an algebra of subsets of $\mathcal{B}$, but not a sigma-algebra if $\mathcal{B}$ is infinite dimensional. A cylindrical measure on $\mathcal{B}$ is a finite finitely-additive, non-negative measure μ on the algebra

$\mathcal{R}(\mathcal{B})$ such that for each Y in $\mathcal{F}$, the restriction $\mu\big|_{\mathcal{R}_Y}$ is sigma-additive. The Fourier transform of μ is the function $\widehat{\mu} : \mathcal{B}^* \to \mathbb{C}$ defined by

$$\widehat{\mu}(y) = \int\limits_{\mathcal{B}} e^{-i\langle x,y\rangle}\, d\mu(x).$$

This integral is with respect to a sigma-additive measure, since the function $x \mapsto e^{-i\langle x,y\rangle}$ is bounded and $\mathcal{R}_{\{y\}}$-measurable and the restriction $\mu\big|_{\mathcal{R}_{\{y\}}}$ is sigma-additive. A cylindrical measure μ is called gaussian if for any $Y \in \mathcal{F}$, the Borel measure

$$\mu_Y(B) = \mu(\{x \in \mathcal{B} : (\langle x, y_1\rangle, \dots, \langle x, y_n\rangle) \in B\})$$

on $\mathbb{R}^n$ is gaussian.

Let $\mathcal{S}(\mathcal{B})$ be the set of bounded linear operators $T : \mathcal{B}^* \to \mathcal{B}$ satisfying the conditions

$$\langle Tx, y\rangle = \langle x, Ty\rangle \ \text{ for each } x, y \in \mathcal{B}^*, \tag{45}$$

$$\langle Tx, x\rangle \geqslant 0 \ \text{ for each } x \in \mathcal{B}^*. \tag{46}$$

It is well-known, see for instance [5], Corollary 1.2, p. 901, that for any $T \in \mathcal{S}(\mathcal{B})$ there exists a unique Gaussian cylindrical measure μ_T on $\mathcal{B}$ such that $\widehat{\mu_T}(x) = e^{-\frac{1}{2}\langle Tx,x\rangle}$ for any $x \in \mathcal{B}^*$. In this case the operator T is called the *covariance operator* of μ. We need the following characterization of σ-additivity of Gaussian measures on ℓ_p. The following theorem can be found in [30].

Theorem **V.** *Let $1 \leqslant p < \infty$ and μ be a gaussian cylindrical measure on the real Banach space $\ell_p(\Lambda)$. Then μ is σ-additive if and only if*

$$\sum_{\alpha\in\Lambda} |m_\alpha|^p < \infty \quad \textit{and} \quad \sum_{\alpha\in\Lambda} s_\alpha^{p/2} < \infty, \quad \textit{where}$$

$$m_\alpha = \int\limits_{\ell_p(\Lambda)} \langle x, e_\alpha\rangle\, d\mu(x) \ \text{ and } \ s_\alpha = \int\limits_{\ell_p(\Lambda)} \langle x, e_\alpha\rangle^2\, d\mu(x).$$

Note that finiteness of the integrals defining s_α imply convergence of integrals defining m_α. One can easily verify that for $\mu = \mu_T$ with $T \in \mathcal{S}(\ell_p(\Lambda))$, $m_\alpha = 0$ and $s_\alpha = \langle Te_\alpha, e_\alpha\rangle$. Thus, Theorem V implies the following corollary.

Corollary **5.8.** *Let $1 \leqslant p < \infty$ and $T \in \mathcal{S}(\ell_p(\Lambda))$. Then μ_T is σ-additive if and only if*

$$\sum_{\alpha\in\Lambda} \langle Te_\alpha, e_\alpha\rangle^{p/2} < \infty.$$

We need the following two lemmas, in which Λ is a countable infinite set and the spaces $\ell_p(\Lambda)$ are assumed to be **real**.

Lemma **5.9.** *Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, $k \in \mathbb{N}$, $A \in \mathcal{S}(\ell_q(\Lambda))$ be such that $\sum\limits_{\alpha\in\Lambda} \langle Ae_\alpha, e_\alpha\rangle^{q/2} < \infty$ and $\{u_n\}_{n\in\mathbb{N}_0}$ be a sequence of vectors from $\ell_p(\Lambda)$ such that $\langle Au_n, u_n\rangle \geqslant 1$ for each $n \in \mathbb{N}_0$. Then for any sequence $a = \{a_n\}_{n\in\mathbb{N}_0}$ of non-negative numbers such that $a \in \ell_k$, there exist $g_1, \dots, g_k \in \ell_q(\Lambda)$ for which*

$$\max_{1\leqslant j\leqslant k} |\langle u_n, g_j\rangle| \geqslant a_n \quad \textit{for any} \ \ n \in \mathbb{N}_0.$$

P r o o f. Without loss of generality we can assume that $\langle Au_n, u_n\rangle = 1$ for each $n \in \mathbb{N}_0$. Indeed, if it is not the case, we can replace u_n by $\langle Au_n, u_n\rangle^{-1/2} u_n$.

Let $K = \{1, \dots, k\}$. For $j \in K$ and $r \in (1, \infty)$ consider the natural projections $P_{r,j} : \ell_r(K \times \Lambda) \to \ell_r(\Lambda)$ and natural embeddings $J_{r_j} : \ell_r(\Lambda) \to \ell_r(K \times \Lambda)$ defined on the canonical basis as $P_{r,j} e_{l,\alpha} = e_\alpha$ and $J_{r,j} e_\alpha = e_{j,\alpha}$. Consider the bounded linear operator $T : \ell_p(K \times \Lambda) \to \ell_q(K \times \Lambda)$ defined by the formula

$$Tx = \sum_{j=1}^{k} J_{q,j} A P_{p,j} x.$$

In other words T is the direct sum of k copies of A. Clearly $T \in \mathcal{S}(\ell_q(K \times \Lambda))$.

Since $\langle Te_{j,\alpha}, e_{j,\alpha}\rangle = \langle Ae_\alpha, e_\alpha\rangle$ for each $(j, \alpha) \in K \times \Lambda$, we observe that

$$\sum_{(j,\alpha)\in K\times\Lambda} \langle Te_{j,\alpha}, e_{j,\alpha}\rangle^{q/2} = k \sum_{\alpha\in\Lambda} \langle Ae_\alpha, e_\alpha\rangle^{q/2} < \infty.$$

By Corollary 5.8 the gaussian cylindrical measure $\mu = \mu_T$ on $\ell_q(K \times \Lambda)$ is σ-additive and therefore extends to a Borel probability measure: the measure of the entire space is 1 since the Fourier transform takes value one at zero.

Let also $u_{j,n} = J_{p,j} u_n \in \ell_p(K \times \Lambda)$, for $j \in K$, $n \in \mathbb{N}_0$. One can easily verify that

$$\langle Tu_{j,n}, u_{l,n}\rangle = \delta_{j,l} \text{ for any } l, j \in K \text{ and } n \in \mathbb{N}_0, \tag{47}$$

where $\delta_{j,l}$ is the Kronecker delta. We take $c > 0$ and consider

$$B_{n,c} = \Big\{ y \in \ell_q(K \times \Lambda) : \sum_{j=1}^{k} |\langle y, u_{j,n}\rangle|^2 \leqslant c^2 a_n^2 \Big\}.$$

We shall estimate $\mu(B_{n,c})$. Consider the Borel probability measure ν on $\mathbb{R}^k$ defined as

$$\nu(B) = \mu\{y \in \ell_q(K \times \Lambda) : (\langle y, u_{1,n}\rangle, \dots, \langle y, u_{k,n}\rangle) \in B\}.$$

From (47), the equality $\widehat{\mu}(z) = e^{-\frac{1}{2}\langle Tz,z\rangle}$ and the definition of ν, it follows that the Fourier transform of ν is $\widehat{\nu}(t) = e^{-|t|^2/2}$. Hence, ν has the density $\rho_\nu(s) = (2\pi)^{-k/2} e^{-|s|^2/2}$. Denote $D_b^k = \{x \in \mathbb{R}^k : |x| \leqslant b\}$. Then

$$\mu(B_{n,c}) = \nu(D^k_{ca_n}) = (2\pi)^{-k/2} \int_{D^k_{ca_n}} e^{-|s|^2/2}\, ds < (2\pi)^{-k/2} \lambda_k(D^k_{ca_n}) = v_k c^k a_n^k,$$

where λ_k is the Lebesgue measure on $\mathbb{R}^k$ and $v_k = (2\pi)^{-k/2} \lambda_k(D_1^k)$. Hence,

$$\mu\left(\bigcup_{n=0}^{\infty} B_{n,c}\right) \leqslant \sum_{n=0}^{\infty} \mu(B_{n,c}) < v_k c^k \sum_{n=0}^{\infty} a_n^k.$$

Since $a \in \ell_k$, by taking c small enough we can ensure that

$$\mu(\Lambda_c) < 1 = \mu(\ell_q(K \times \Lambda)), \quad \text{where } \Lambda_c = \bigcup_{n=0}^{\infty} B_{n,c}.$$

Therefore, there must be $y \in \ell_q(K \times \Lambda) \setminus \Lambda_c$. Clearly,

$$\sum_{j=1}^{k} |\langle P_{q,j} y, u_n \rangle|^2 = \sum_{j=1}^{k} |\langle y, u_{j,n} \rangle|^2 > c^2 \, a_n^2, \quad \text{for} \ \ n \in \mathbb{N}_0.$$

Hence $\max\limits_{1 \leqslant j \leqslant k} |\langle g_j, u_n \rangle| > a_n$ for each $n \in \mathbb{N}_0$, where $g_j = (\sqrt{k}/c) P_{q,j} y \in \ell_q(\mathbb{Z})$. $\square$

Lemma **5.10.** *Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, and $\{x_n\}_{n \in \mathbb{N}_0}$ be a sequence in $\ell_p(\Lambda)$, satisfying (38) with $0 < a < \min\{2, q\}$. Then there exist $k \in \mathbb{N}$ and $g_1, \dots, g_k \in \ell_q(\Lambda)$ such that*

$$\max_{1 \leqslant j \leqslant k} |\langle x_n, g_j \rangle| \geqslant 1 \ \ \textit{for any} \ \ n \in \mathbb{N}_0.$$

Proof. Denote $d = \max\{a, 2a/q\}$. Since $a < \min\{2, q\}$, we see that $d < 2$ and we can choose $k \in \mathbb{N}$ such that $k(1 - d/2) \geqslant a$. Let $s_n = \|x_n\|_p^{-d}$, $a_n = \|x_n\|_p^{d/2-1}$ and $u_n = \|x_n\|_p^{d/2-1} x_n$. From (38) it follows that

$$\sum_{n=0}^{\infty} s_n^r < \infty, \quad \text{where} \ \ r = \min\{1, q/2\} \tag{48}$$

and that $\{a_n\}_{n \in \mathbb{N}_0} \in \ell_k$. By Hahn–Banach theorem, for any $n \in \mathbb{N}_0$, we can choose $y_n \in \ell_q(\Lambda)$ such that $\|y_n\|_q = 1$ and $\langle x_n, y_n \rangle = \|x_n\|_p$. Consider the operator

$$A : \ell_p(\Lambda) \to \ell_q(\Lambda), \qquad Ax = \sum_{n=0}^{\infty} s_n \langle x, y_n \rangle y_n.$$

According to (48) the sequence $\{s_n\}$ is summable and therefore the operator A is bounded. One can easily verify that the conditions (45) and (46) for A are satisfied. Hence $A \in \mathcal{S}(\ell_q(\Lambda))$. Clearly

$$\langle Au_n, u_n \rangle = \|x_n\|_p^{d-2} \sum_{m=0}^{\infty} s_m \langle x_n, y_m \rangle^2 \geqslant s_n \|x_n\|_p^{d-2} \langle x_n, y_n \rangle^2 = \|x_n\|_p^{-d} \|x_n\|_p^{d-2} \|x_n\|_p^2 = 1.$$

We shall check now that

$$\sum_{\alpha \in \Lambda} \langle Ae_\alpha, e_\alpha \rangle^{q/2} < \infty. \tag{49}$$

For any $n \in \mathbb{N}_0$ consider the sequence z_n of non-negative numbers with the index set Λ defined by the formula $\langle z_n, e_\alpha \rangle = \langle y_n, e_\alpha \rangle^2$. Let also z be the sequence defined as $\langle z, e_\alpha \rangle = \langle Ae_\alpha, e_\alpha \rangle$. Since for any $\alpha \in \Lambda$,

$$\langle z, e_\alpha \rangle = \langle Ae_\alpha, e_\alpha \rangle = \sum_{n=0}^{\infty} s_n \langle y_n, e_\alpha \rangle^2 = \sum_{n=0}^{\infty} s_n \langle z_n, e_\alpha \rangle,$$

we see that $z = \sum\limits_{n=0}^{\infty} s_n z_n$ in the coordinatewise convergence sense.

Case $p \leqslant 2$. In this case $q \geqslant 2$. Clearly $z_n \in \ell_{q/2}(\Lambda)$ and $\|z_n\|_{q/2} = \|y_n\|_q = 1$ for each $n \in \mathbb{N}_0$. By (48) the sequence s_n of positive numbers is summable and therefore the series $\sum\limits_{n=0}^{\infty} s_n z_n$ is absolutely convergent in the Banach space $\ell_{q/2}(\Lambda)$. Hence $z \in \ell_{q/2}(\Lambda)$ and (49) follows.

Case $p > 2$. In this case $q < 2$.

Recall that for $0 < \rho < 1$, the space $\ell_\rho(\Lambda)$ of sequences $x = \{x_\alpha\}_{\alpha\in\Lambda}$ in $\ell_\infty(\Lambda)$ for which

$$\pi_\rho(x) = \sum_{\alpha\in\Lambda} |x_\alpha|^\rho < \infty$$

is no longer a Banach space. The function π_ρ is a pseudonorm, which turns $\ell_\rho(\Lambda)$ into a complete metrizable topological vector space, which is not locally convex. The pseudonorm π_ρ satisfies the triangle inequality $\pi_\rho(x + y) \leqslant \pi_\rho(x) + \pi_\rho(y)$ and the homogeneity condition $\pi_\rho(cx) = c^\rho \pi_\rho(x)$ for $c \in \mathbb{R}$ and $x, y \in \ell_\rho(\mathbb{Z})$.

Clearly $z_n \in \ell_{q/2}(\Lambda)$ and $\pi_{q/2}(t_n) = \|y_n\|_q^q = 1$ for each $n \in \mathbb{N}_0$. By (48), we have $\sum\limits_{n=0}^{\infty} s_n^{q/2} < \infty$. From the triangle inequality and homogeneity of $\pi_{q/2}$ it follows that the series $\sum\limits_{n=0}^{\infty} s_n z_n$ is convergent in the space $\ell_{q/2}(\Lambda)$ and therefore $z \in \ell_{q/2}(\lambda)$. Hence (49) is satisfied.

Thus, in any case all conditions of Lemma 5.9 are fulfilled. Hence there exist $g_1, \dots, g_k \in \ell_q(\mathbb{Z})$ such that $\max\limits_{1\leqslant j\leqslant k} |\langle u_n, g_j\rangle| \geqslant a_n$ for any $n \in \mathbb{N}_0$. Therefore

$$\max_{1\leqslant j\leqslant k} |\langle x_n, g_j\rangle| = \|x_n\|_p^{1-d/2} \max_{1\leqslant j\leqslant k} |\langle u_n, g_j\rangle| \geqslant \|x_n\|_p^{1-d/2} a_n = 1 \quad \text{for any} \quad n \in \mathbb{N}_0. \quad \square$$

Lemma **5.11.** *Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, and $\{x_n\}_{n\in\mathbb{N}_0}$ be a sequence in the real or complex Banach space $\ell_p(\Lambda)$, satisfying* (38) *with $0 < a < \min\{q, 2\}$. Then zero is not in the weak closure of $\{x_n : n \in \mathbb{N}_0\}$.*

Proof. The real case follows immediately from Lemma 5.10. In the complex case it suffices to notice that the complex Banach space $\ell_p(\Lambda)$, considered as a real one, is isomorphic to the real Banach space $\ell_p(\Lambda)$. $\square$

We are ready to prove Proposition 5.3. Let $y \in \ell_p(\Lambda) \setminus S$. Applying Lemma 5.11 to the sequence, $\{x_n - y\}_{n\in\mathbb{N}_0}$, we see that zero is not in the weak closure of $\{x_n - y : n \in \mathbb{N}_0\}$. Hence y is not in the weak closure of S. Thus S is weakly closed. The proof is complete.

6 Concluding remarks and open problems

We start with a few general remarks. Since the Banach space ℓ_1 enjoys the Schur property: weak and norm convergence of sequences are equivalent [10], weak sequential supercyclicity and weak sequential hypercyclicity of bounded linear operators on ℓ_1 are equivalent to supercyclicity and hypercyclicity respectively. For operators acting on general Banach spaces it is not true, as follows from the example of Bayart and Matheron [6]. Next proposition shows that it is true for operators on general Banach spaces under the additional condition that there exists a compact operator with dense range, commuting with the given one.

Proposition **6.1.** *Let T be a bounded linear operator acting on a Banach space $\mathcal{B}$. Assume that there is a compact operator S, acting on $\mathcal{B}$, such that S has dense range and $TS = ST$. Then T is weakly sequentially supercyclic if and only if T is supercyclic and T is weakly sequentially hypercyclic if and only if T is hypercyclic.*

In order to prove Proposition 6.1 we need the following topological lemma.

Lemma **6.2.** *Let X and Y be topological spaces and $S : X \to Y$ be a sequentially continuous map with sequentially dense range. Let also $A \subset X$ be a sequentially dense subset of X. Then $S(A)$ is sequentially dense in Y.*

Proof. Let $M = [S(A)]_{\rm seq}$ be the sequential closure of $S(A)$ in Y. Since S is sequentially continuous and M is sequentially closed, we see that $S^{-1}(M)$ is sequentially closed in X. Since $A \subseteq S^{-1}(M)$ and A is sequentially dense in X, we have $X = S^{-1}(M)$. Hence $S(X) \subseteq M$. Since $S(X)$ is sequentially dense in Y, and M is sequentially closed in Y, we obtain $M = Y$. □

Proof of Propositions 6.1 Let $x \in \mathcal{B}$ be a weakly sequentially supercyclic vector for T. Since S is compact, it is sequentially continuous as a map from $(\mathcal{B}, \sigma)$ to $(\mathcal{B}, \tau)$, where σ and τ stand for the weak topology and norm topologies respectively [23]. Since τ is metrizable, we have that the range of S is sequentially dense in $(\mathcal{B}, \tau)$. Since $O_{\rm pr}(T, x)$ is sequentially dense in $(\mathcal{B}, \sigma)$, Lemma 6.2 implies that $S(O_{\rm pr}(T, x))$ is sequentially dense in $(\mathcal{B}, \tau)$ and therefore norm-dense in $\mathcal{B}$. Taking into account that T and S commute we obtain that $S(O_{\rm pr}(T, x)) = O_{\rm pr}(T, Sx)$ and therefore the projective orbit $O_{\rm pr}(T, Sx)$ is norm dense in $\mathcal{B}$. Thus, Sx is a supercyclic vector for T. The proof of the hypercyclicity case is exactly the same. One has just to consider the orbits instead of the projective orbits. □

Proposition 6.1 leads to some interesting questions.

QUESTION **6.3**. *Is it possible in Proposition* 6.1 *to replace weak sequential supercyclicity or hypercyclicity by weak supercyclicity or hypercyclicity? In particular, does there exist a non-supercyclic weakly supercyclic compact operator?*

Bes, Chan and Sanders [7] asked whether there exists a weakly 1-sequentially hypercyclic operator which is not norm hypercyclic. The question remains open as well as the following ones.

QUESTION **6.4**. *Does there exist a non-hypercyclic weakly sequentially hypercyclic operator?*

QUESTION **6.5**. *Does there exist a weakly sequentially hypercyclic operator which is not weakly* 1*-sequentially hypercyclic?*

QUESTION **6.6**. *Does there exist a weakly sequentially supercyclic operator which is not weakly* 1*-sequentially supercyclic?*

Finally observe that according to Proposition 1.1, Theorem 1.2 provides an example of a weakly supercyclic antisupercyclic operator on a Hilbert space, which answers a question raised in [29].

6.1 Measures

The construction of a measure in the proof of Theorem 1.2 does not provide any control of the rate of decaying of the Fourier coefficients. In principle it is possible to make an effective version of the construction, but one thing is obvious: the Fourier coefficients tend to zero extremely slowly. This motivates the following question.

QUESTION **6.7**. *Does there exist any condition on the rate of the Fourier coefficients $\widehat{\mu}(n)$ of a Borel probability measure on $\mathbb{T}$ (weaker then the trivial one: $\sum |\widehat{\mu}(n)|^2 < \infty$) implying that the multiplication operator $Mf(z) = zf(z)$ acting on $L_2(\mu)$ is not weakly supercyclic?*

On the other hand, it would be desirable to find simpler measures, satisfying the assertions of Theorem 1.2.

QUESTION **6.8**. *Does there exist $\mu \in \mathcal{M}_0 \cap \mathcal{P}$ being an infinite convolution of a sequence of discrete probability measures, such that the multiplication operator $Mf(z) = zf(z)$ acting on $L_2(\mu)$ is weakly supercyclic? What about self-similar measures?*

As it was remarked by Bayart and Matheron [6], if the operator $Mf(z) = zf(z)$ acting on $L_2(\mu)$ with $\mu \in \mathcal{M}_+$ is weakly supercyclic, then μ is singular. In particular, the measure in Theorem 1.2 is singular. It follows from the fact that if $\mu \in \mathcal{M}_+$ is not singular, that is μ has a non-trivial absolutely continuous component, then there exists $n \in \mathbb{N}$ such that the operator M^n is not cyclic, while the powers of any weakly supercyclic operator are weakly supercyclic and

therefore cyclic. It is not, however, the feature of absolute continuity since M^3 is not cyclic if M acts on $L_2(\mu)$, where μ is the standard Cantor measure, which is purely singular.

On the other hand if A is a Borel measurable subset of $\mathbb{T}$ such that $z^n \neq w^n$ for any $n \in \mathbb{N}$ and any different $z, w \in A$ and $\mu \in \mathcal{M}_+ \cap \mathcal{M}(A)$, then M^n is cyclic for any $n \in \mathbb{N}$. It follows from the observation that in this case for any $n \in \mathbb{N}$ there exists $\mu^n \in \mathcal{M}_+$ such that the operator M^n acting on $L_2(\mu)$ is unitarily equivalent to M acting on $L_2(\mu^n)$. Observe that the above property of A is strictly weaker than independence of A. This leads us to the following question.

Question **6.9**. *Let $\mu \in \mathcal{P} \cap \mathcal{M}_0$ be such that* $\operatorname{supp}(\mu)$ *is independent. Is M acting on $L_2(\mu)$ weakly supercyclic?*

It worth noting that the class of measures under the hypothesis of Question 6.9 is quite large. For instance, for any Borel measurable set $A \subset \mathbb{T}$ such that the set $\mathcal{P} \cap \mathcal{M}_0 \cap \mathcal{M}(A)$ is non-empty, there exists a measure $\mu \in \mathcal{P} \cap \mathcal{M}_0$, whose support is an independent subset of A, see [17].

6.2 Bilateral weighted shifts

Theorem 1.6 together with Theorem S characterizes weakly supercyclic bilateral weighted shifts on $\ell_p(\mathbb{Z})$ with $p \leqslant 2$ and weakly hypercyclic bilateral weighted shifts on $\ell_p(\mathbb{Z})$ with $p < 2$. Proposition 3.4 provides a sufficient condition of weak supercyclicity and weak hypercyclicity of bilateral weighted shifts on general $\ell_p(\mathbb{Z})$. It is not clear whether the condition of Proposition 3.4 is also necessary. This leads to the following problem.

Problem **6.10**. *Characterize* (*in terms of weight sequences*) *weakly supercyclic bilateral weighted shifts on $\ell_p(\mathbb{Z})$ for $p > 2$ and weakly hypercyclic bilateral weighted shifts on $\ell_p(\mathbb{Z})$ for $p \geqslant 2$.*

Note also that Proposition 3.4 provides more than just a weakly supercyclic or a weakly hypercyclic vector x for a bilateral weighted shift T. Namely, it ensures that $\{\lambda T^{r_n}x : n \in \mathbb{N}_0, \lambda \in \mathbb{C}\}$ or $\{T^{r_n}x : n \in \mathbb{N}_0\}$ are weakly dense for an exponentially growing sequence of $\{r_n\}$ of positive integers. Indeed, condition (W2) of Proposition 3.4 implies that $\varliminf\limits_{n\to\infty} (r_n)^{1/n} \geqslant \frac{\sqrt{5}+1}{2} > 1$. One way to approach Problem 6.4 could be to find out whether there exists a weakly supercyclic or a weakly hypercyclic bilateral weighted shift T such that the sets of the shape $\{\lambda T^{r_n}x : n \in \mathbb{N}_0, \lambda \in \mathbb{C}\}$ or $\{T^{r_n}x : n \in \mathbb{N}_0\}$ are not weakly dense for any exponentially growing sequence of $\{r_n\}$ of positive integers.

Using Proposition 3.4 and the technique of the proof of Theorem 1.6 it is possible for any $p \geqslant 2$ to find a bilateral weighted shift, which is weakly hypercyclic on $\ell_p(\mathbb{Z})$ and not weakly hypercyclic on $\ell_r(\mathbb{Z})$ for each $r < p$. Thus, the infinum of p's for which a given bilateral weighted shift is weakly hypercyclic on $\ell_p(\mathbb{Z})$ is a parameter taking all values between 2 and ∞. Thus, any characterization of hypercyclic bilateral weighted shifts on $\ell_p(\mathbb{Z})$ for $p \geqslant 2$ must depend on the parameter p.

6.3 Tightness of Propositions 5.2–5.4

The following theorem is known as Dvoretzky theorem on almost spherical sections [12]. Somewhat weaker version of this theorem was obtained earlier by Dvoretzky and Rogers [13].

Theorem **D.** *For each $n \in \mathbb{N}$ and each $\varepsilon > 0$, there exists $m = m(n, \varepsilon) \in \mathbb{N}$ such that for any Banach space $\mathcal{B}$ with $\dim \mathcal{B} \geqslant m$ there is an n-dimensional linear subspace L in $\mathcal{B}$ and a basis $e_1, \dots, e_n$ in L for which*

$$\left\| \sum_{j=1}^{n} c_j e_j \right\|_{\mathcal{B}} \leqslant \left(\sum_{j=1}^{n} |c_j|^2 \right)^{1/2} \leqslant (1+\varepsilon) \left\| \sum_{j=1}^{n} c_j e_j \right\|_{\mathcal{B}} \quad \text{for any } (c_1, \dots, c_n) \in \mathbb{C}^n.$$

We use this theorem in order to prove the following proposition, which allows us to demonstrate tightness of Propositions 5.2–5.4.

PROPOSITION **6.11.** *For any infinite dimensional Banach space $\mathcal{B}$ and any sequence $\{c_n\}_{n\in\mathbb{N}_0}$ of positive numbers such that $\sum\limits_{n=0}^{\infty} c_n^{-2}=\infty$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{B}$ such that $\|x_n\|=c_n$ for each $n\in\mathbb{N}_0$ and zero is in the weak closure of $\{x_n : n\in\mathbb{N}_0\}$.*

Proof. Pick a strictly increasing sequence $\{n_k\}_{k\geqslant 0}$ of integers such that $n_0=0$ and

$$\lim_{k\to\infty}\sum_{j=n_{k-1}}^{n_k-1} c_j^{-2}=\infty. \tag{50}$$

Denote $j_k=n_k-n_{k-1}$, $k\in\mathbb{N}$. By Theorem D, for each $k\in\mathbb{N}$, there exist a linear subspace F_k of $\mathcal{B}$ with $\dim F_k=j_k$ and a basis $e_{n_{k-1}},\dots,e_{n_k-1}$ in F_k such that

$$\Big\|\sum_{j=n_{k-1}}^{n_k-1} c_je_j\Big\|_{\mathcal{B}}\leqslant\Big(\sum_{j=n_{k-1}}^{n_k-1}|c_j|^2\Big)^{1/2}\leqslant 2\Big\|\sum_{j=n_{k-1}}^{n_k-1} c_je_j\Big\|_{\mathcal{B}} \tag{51}$$

for any complex numbers c_j. In what follows, we assume that F_k's carry the norm inherited from $\mathcal{B}$. The inequality for the dual norm reads as follows

$$\frac12\|f\|_{F_k^*}\leqslant\Big(\sum_{j=n_{k-1}}^{n_k-1}|\langle f,e_j\rangle|^2\Big)^{1/2}\leqslant\|f\|_{F_k^*}\ \ \text{for each}\ \ f\in F_k^*. \tag{52}$$

Denote $x_n=c_ne_n/\|e_n\|$ for $n\in\mathbb{N}_0$. Obviously $\|x_n\|=c_n$. It remains to prove that zero is in the weak closure of $\{x_n : n\in\mathbb{N}_0\}$. Suppose the contrary. Then there exist $g_1,\dots,g_m\in\mathcal{B}^*$ such that

$$\max_{1\leqslant l\leqslant m}|\langle g_l,x_n\rangle|\geqslant 1\ \ \text{for each}\ n\in\mathbb{N}_0. \tag{53}$$

Denote $M=\max\limits_{1\leqslant l\leqslant m}\|g_j\|_{\mathcal{B}^*}$ and for each positive integer k let $h_l^k\in F_k^*$ be the restriction of g_l to F_k. From (51) it follows that $\|e_n\|\geqslant 1/2$ for each n. If $1\leqslant l\leqslant m$ and $n_{k-1}\leqslant n\leqslant n_k-1$, then

$$|\langle h_l^k,e_n\rangle|=|\langle g_l,e_n\rangle|=\|e_n\|c_n^{-1}|\langle g_l,x_n\rangle|\geqslant(2c_n)^{-1}|\langle g_l,x_n\rangle|.$$

Using (53) and the last display, we obtain

$$\sum_{l=1}^m|\langle h_l^k,e_n\rangle|^2\geqslant(2c_n)^{-2}\ \ \text{for}\ \ n_{k-1}\leqslant n\leqslant n_k-1.$$

Taking (52) into account, we get

$$\sum_{l=1}^m\|h_l^k\|^2_{F_k^*}\geqslant\sum_{l=1}^m\sum_{j=n_{k-1}}^{n_k-1}|\langle h_l^k,e_j\rangle|^2\geqslant\frac14\sum_{j=n_{k-1}}^{n_k-1}c_j^{-2}.$$

Since $\|h_l^k\|_{F_k^*}\leqslant\|g_l\|_{\mathcal{B}^*}\leqslant M$, we see that $4mM^2\geqslant\sum\limits_{j=n_{k-1}}^{n_k-1}c_j^{-2}$ for any positive integer k, which contradicts (50). □

Corollary **6.12.** *Let $1 \leqslant p \leqslant \infty$, $\mathcal{B}_p = \ell_p$ if $1 \leqslant p < \infty$ and $\mathcal{B}_\infty = c_0$ and $\{c_n\}_{n\in\mathbb{N}_0}$ be a sequence of positive numbers such that $\sum\limits_{n=0}^{\infty} c_n^{-r} = \infty$, where $r = \min\{2, q\}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{B}$ such that $\|x_n\| = c_n$ for each $n \in \mathbb{N}_0$ and zero is in the weak closure of $\{x_n : n \in \mathbb{N}_0\}$.*

Proof. The case $1 \leqslant p \leqslant 2$ follows from Proposition 6.11. If $p > 2$, we can take $x_n = c_n e_n$ and apply Lemma 2.2. □

Corollary 6.12 for $p = 2$ and $p = \infty$ implies that conditions on the growth of $\|x_n\|$ in Propositions 5.2 and 5.4 are best possible. Proposition 5.3 and Corollary 6.12 lead to the natural conjecture that the best possible condition on the growth of $\|x_n\|$ implying weak closeness of $\{x_n : n \in \mathbb{N}_0\}$ in ℓ_p is (38) with $a = \min\{2, q\}$. In order to prove this conjecture it would suffice to answer the following question affirmatively.

Question **6.13**. *Let $1 \leqslant p < \infty$, $\{x_n\}_{n\in\mathbb{N}_0}$ be a sequence in the unit sphere of a the complex Banach space ℓ_p and $\{s_n\}_{n\in\mathbb{N}_0}$ be a sequence of positive numbers such that $\sum\limits_{n=0}^{\infty} s_n^r = 1$, where $r = \min\{2, q\}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Does there exist $y \in \ell_q$ such that $\|y\|_q = 1$ and $|\langle x_n, y\rangle| \geqslant s_n$ for each $n \in \mathbb{N}_0$?*

Note that an affirmative answer to this question would also provide an interesting generalization of Ball's theorem (Theorem B1) and possibly lead to further applications in convex analysis.

6.4 Sequential weak topology

In this final section we discuss the nature of weak sequential density and thus of weak sequential supercyclicity and hypercyclicity. Recall that a topological space (X, τ) is called *sequential* if a subset of X is closed if and only if it is sequentially closed. A subset A of a topological vector space (X, τ) is called sequentially open if $X \setminus A$ is sequentially closed. It is straightforward to verify that the collection τ_{seq} of sequentially open subsets of a topological space (X, τ) forms a topology. Moreover, $\tau \subseteq \tau_{\text{seq}}$ and (X, τ_{seq}) is sequential and a sequence converges in (X, τ) if and only if it converges to the same limit in (X, τ_{seq}).

For a Banach space $\mathcal{B}$, $\sigma = \sigma(\mathcal{B}, \mathcal{B}^*)$ stands for the weak topology of $\mathcal{B}$ and σ_{seq} stands for the corresponding sequential topology. From the above it follows that a set $A \subseteq \mathcal{B}$ is weakly sequentially dense in $\mathcal{B}$ if and only if it is dense in σ_{seq}. Thus, the concepts of weak sequential hypercyclicity and supercyclicity (unlike weak 1-sequential hypercyclicity and supercyclicity) are topological. Namely they are just hypercyclicity and supercyclicity with respect to the topology σ_{seq} intermediate between the weak and the norm topologies.

Finally we make a few remarks on the nature of the topology σ_{seq}. From the Schur Theorem [10] it follows that the topology σ_{seq} on the Banach space ℓ_1 coincides with the norm topology. In [22] it is observed that there are Banach spaces $\mathcal{B}$ for which $(\mathcal{B}, \sigma_{\text{seq}})$ fails to be a topological vector space: the addition $(x, y) \mapsto x + y$, although being separately continuous, may fail to be continuous. It is also demonstrated in [22] that if $\mathcal{B}^*$ is separable then σ_{seq} coincides with the so-called bounded weak topology, which is the strongest topology that agrees with the weak topology on the bounded sets. According to the Banach–Dieudonné theorem, see for instance [28], the bounded weak topology on a reflexive Banach space coincides with the pre-compact convergence topology, that is the topology of uniform convergence over the norm pre-compact subsets of $\mathcal{B}^*$. It worth mentioning that $\mathcal{B}$ with the pre-compact convergence topology is a complete locally convex topological vector space. For a characterization of local convexity of the bounded weak topology we refer to [16]. Thus, we have the following

Proposition **6.14.** *Let $\mathcal{B}$ be a separable reflexive Banach space. Then the weak sequential topology $\sigma_{\rm seq}$ on $\mathcal{B}$ coincides with the pre-compact convergence topology.*

According to this proposition weak sequential supercyclicity and hypercyclicity of bounded linear operators on a separable reflexive Banach space are exactly supercyclicity and hypercyclicity with respect to the pre-compact convergence topology. Note that for infinite dimensional Banach spaces the pre-compact convergence topology is strictly stronger than the weak topology and strictly weaker than the norm topology.

Acknowledgements. Partially supported by Plan Nacional I+D+I Grant BFM2003-00034, Junta de Andalucía FQM-260 and British Engineering and Physical Research Council Grant GR/T25552/01. The author is grateful to Alfonso Montes–Rodríguez and Frédéric Bayart for many helpful comments.

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